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LONGITUDINALLY EXCITED NONLINEAR LIQUID MOTION IN A CIRCULAR CYLINDRICAL TANK WITH ELASTIC BOTTOM

A THESIS

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LONGITUDINALLY EXCITED NONLINEAR LIQUID MOTION

IN A CIRCULAR CYLINDRICAL TANK WITH ELASTIC BOTTOM

Approved: ______________________

Chairman ______________________

Date approved by Chairman: 12/12/69

To

my Mother and Grace

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TABLE OF CONTENTS

LIST OF ILLUSTRATIONS

LIST OF ILLUSTRATIONS (Concluded)

LIST OF SYMBOLS

a	radius of the tank
$\overline{h}$	thickness of the tank bottom
H	undisturbed liquid height
p	pressure
$\overline{r}$	ratio of the surface response frequency to the natural frequency
x,y,z r,θ,z	tank fixed coordinates
$\overline{u},\overline{v},\overline{w}$	velocity components of the fluid particle in r,θ,z directions
w	deflection of the tank bottom
$Z(t)$	displacement excitation of the tank
Z_o	amplitude of the displacement excitation
$\eta(r,\theta,t)$	liquid free surface
η^*	average free surface amplitude
$a_{mn}(t)$	time function corresponding to the $(m,n)^{th}$ mode appearing in the Fourier-Bessel expansion of the liquid free surface ($a_n(t)$ corresponds to the case of axisymmetric mode)
$\alpha_{mn}(t),\beta_{mn}(t)$	time functions corresponding to the $(m,n)^{th}$ mode appearing in the Fourier-Bessel expansion of the velocity potential ($\alpha_n(t)$, $\beta_n(t)$ correspond to the case of axisymmetric mode)

J_m	Bessel function of the first kind of order m
$A_{mn}(t) = \frac{a_{mn}(t)}{a}$	nondimensional time function appearing in the series expression for the liquid free surface ($A_n(t)$ corresponds to the axisymmetric case)
$\lambda_{mn}a$	Eigenvalue corresponding to the $(m,n)^{th}$ mode ($\lambda_n a$ corresponds to the axisymmetric mode)
ρ_o	liquid mass density
$\bar{\rho}$	tank bottom density per unit area
Φ	velocity potential
ω_{mn}	natural frequency of the $(m,n)^{th}$ mode ($\omega_n(t)$ corresponds to the case of the axisymmetric mode)
Ω	tank excitation frequency

SUMMARY

This dissertation is concerned with the theoretical investigation of liquid motion in a circular cylindrical container subjected to longitudinal excitation of sinusoidal form. The basic equations are derived for an incompressible and nonviscous liquid. The motion of the liquid is considered to be irrotational. The container bottom is treated as a thin elastic plate, while the dynamic and kinematic conditions of the liquid free surface are represented by nonlinear equations. Depending on the excitation parameters, the free surface will exhibit a plane surface or oscillate with a finite amplitude. The finite amplitude can be a one-half subharmonic, harmonic, or superharmonic of the excitation frequency. Both one-half subharmonic and harmonic responses have been treated in more detail. It was found that the influence of the elastic bottom upon the response of the liquid is more significant as the tank diameter increases. Furthermore, as the bottom thickness decreases, the elastic effect is more obvious. The liquid height is also a factor affecting the liquid response. As the liquid height decreases, the influence of the elastic bottom upon the liquid response plays an important role. A comparison of the responses of the liquid in a container with an elastic bottom as well as a rigid bottom has been made.

The free surface elevations and the forces and moments due to the liquid motions are obtained also.

CHAPTER I

INTRODUCTION

In the design of large space vehicles, new problem areas have been discovered, areas which were of minor importance in smaller vehicles. The increase of the length of the space vehicle results in decreasing frequencies while the increase of the diameter of the propellant containers yields lower natural frequencies of the liquid. Both trends will influence the stability of the overall space vehicle considerably.[1] Therefore, the investigation of propellant sloshing problems has attracted the attention of many researchers.

The dynamic behavior of a liquid in a moving container has been extensively investigated by many authors during the last decade. For literature surveys, Abramson[2] and Cooper[3] may be consulted. The liquid motion in a longitudinally excited container, however, has been less closely examined. The reason for this may be the fact that for smaller vehicles the influence of longitudinally excited propellant sloshing exhibits a minor effect upon the overall vehicle behavior and the linear analysis fails to predict the liquid response amplitude for longitudinally forced motion.

Longitudinal excitations of the container in actual space flight may arise from the dynamic coupling between the vehicle structure, engine thrust variations, pump dynamics, the pneumatic tank pressure regulation system and the feedline dynamics (see Figure 1). They form the so called POGO-problem. In the light of this problem, the investigation of the longitudinal oscillations is particularly important in the case of manned launch vehicles because of the possible occurring of the acceleration effects on astronauts, their vision and manual reactions.

When a partially filled container is longitudinally excited along its axis, the liquid free surface may oscillate with finite amplitude or remain a plane, depending on the values of the parameters such as excitation amplitude and forcing frequency. Also the elastic properties of the tank walls contribute to this problem. The essential difference between longitudinally forced excitations and laterally forced excitations is that for certain excitation parameters subharmonic liquid responses exist.

In 1831, Faraday[4] made the first experimental study of liquid motions in a longitudinally excited container. He noticed that the frequency of the liquid oscillations was only one half that of the excitation. Later, in 1868-1870, Mathiessen[5,6] made the same kind of experimental study. He, however, observed only the harmonic oscillation response. The disagreement between the result of Faraday and Mathiessen led Lord Rayleigh to make a further series of experiments which agreed with the observation of Faraday. Finally he concluded that Mathiessen's results were in error. In 1954, Benjamin and Ursell[7] made theoretical investigations of the stability of the plane free surface when the container was subjected to longitudinal excitations. Their linear analysis led to the Mathieu equation and predicted that the plane free surface is unstable when the frequency of any mode of the free standing wave is one-half n times the forcing frequency, where n is an integer. Since n can be 1, 2, 3, ... etc., the liquid motion can be one-half subharmonic, harmonic, or superharmonic, they suggested that the experimental results observed by Faraday, Mathiessen, and Rayleigh may all have been correct. Experiments by Benjamin and Ursell confirmed only subharmonic motions.

Utilizing the model of wave motions developed by Penny and Price[8] for the case of free oscillations, Skalak and Yarymovych[9] made the first theoretical investigation of the description of the finite amplitude surface motion for an infinitely deep rectangular tank subjected to longitudinal excitations. Apparently agreeing with Benjamin and Ursell, their experiments yielded only one-half subharmonic liquid motions. In view of the agreement of the nonviscous theory with their experiments, they suggested that damping was of a secondary effect on the one-half subharmonic liquid response.

Using the general method of attack similar to that used by Penny and Price, and Skalak and Yarymovych, a nonlinear analytical and experimental study of liquid motions in a longitudinally excited, finite depth, rigid circular cylindrical container was investigated by Dodge, Kana, and Abramson.[10] Particularly the low-frequency excitation and the corresponding liquid response were emphasized. In their experiments, both one-half subharmonic and harmonic liquid responses were observed. Furthermore, they indicated that the amplitudes of harmonic and superharmonic liquid responses are considerably smaller than those of the one-half subharmonic response, and they concluded that in reality damping prevented the occurrence of the proper harmonic or superharmonic responses.

Using the tank geometry as a parameter, a nonlinear analytical study of the liquid motion in a longitudinally excited cylindrical tank of annular sector cross section was carried out by Woodward.[11] In these investigations the circular cross section is contained as a special case.

For rotating tanks, the linear liquid motion for longitudinal excitation was investigated both analytically and experimentally by Skalak and Conly.[12]

In all of these analyses the liquid containers were considered as rigid. This assumption is by no means true, since the wall thickness of the container is relatively small in comparison with the radius of the container. Therefore, the next question that arises is aimed at the influence of an elastic container bottom upon the response of the liquid.

The interaction between liquid propellant and elastic structure has been studied by many authors. Bhuta and Koval studied the coupled free oscillations of liquid in a rigid circular cylindrical container with a flexible bottom. In their analyses, the bottom was treated as a membrane[13] and as a plate.[14] Taking into account the surface tension and low gravity, Siekmann and Chang[15] investigated the coupled free oscillations of the liquid in a rigid circular cylindrical tank with a flexible bottom. Bauer, Hsu, and Wang[16] treated the coupled free oscillations of a long, elastic rectangular tank and a cylindrical container with elastic walls. The liquid sloshing in an annular cylindrical tank with elastic walls and a rigid bottom was also studied by Bauer, Siekmann, and Wang.[17] The forced longitudinal oscillations of liquid in a rigid circular cylindrical tank with a thin membrane bottom was investigated by Tong and Fung.[18]

All analyses concerned with the hydroelastic problem were based on linear liquid and elastic equations. For a rigid container, however, the behavior of the liquid differs markedly from that predicted by the linearized theory.[10,19] Therefore, a nonlinear analysis of liquid motions in a circular cylindrical container with elastic bottom and being subjected to longitudinal excitation shall be carried out in the following investigation.

For this reason, the basic equations are derived for incompressible and nonviscous liquid. The container bottom is treated as a thin plate, while the dynamic and kinematic free surface conditions of the liquid are represented by nonlinear equations. The analysis shall be restricted to the axisymmetric case and shall present subharmonic and harmonic responses of the system, as well as some of the mechanical values, such as liquid elevation, liquid forces, and moments.

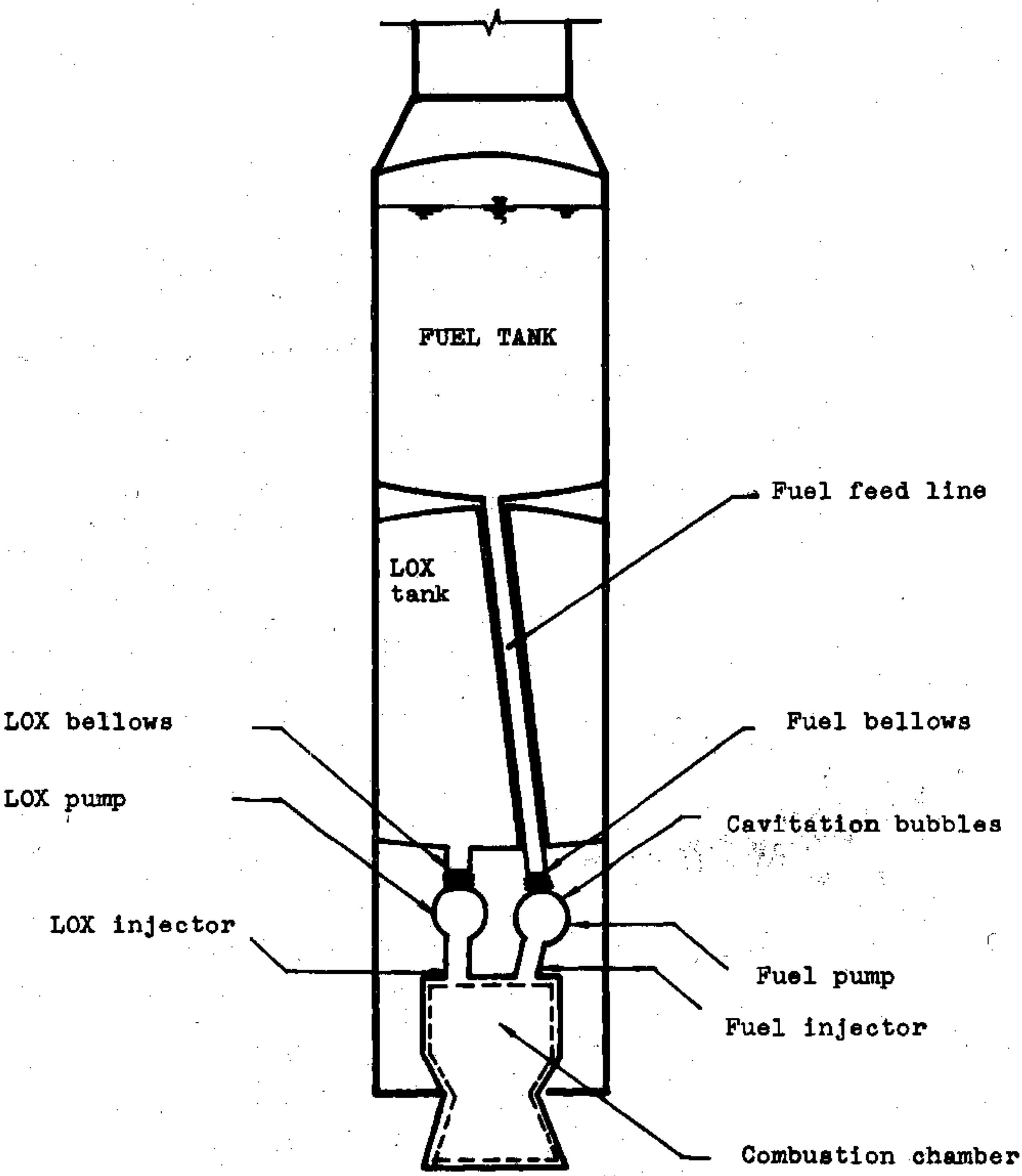


Figure 1. Schematic Representation of a Simple Space Vehicle

CHAPTER II

STATEMENT OF THE PROBLEM

A circular cylindrical container with rigid side walls and elastic bottom contains a liquid with a free surface. The tank walls are subjected to a longitudinal excitation of sinusoidal form as shown in Figure 2. The problem is to determine the motion of the liquid as well as the liquid elevation, and the liquid force and moment for various values of the excitation amplitude Z_0 and forcing frequency Ω.

The liquid is assumed to be inviscid and incompressible. In addition, the motion is considered to be irrotational. The deflection of the elastic bottom from static equilibrium is considered small so that the linear theory of the plate may be applied. The mean free surface of the liquid is assumed to be a plane perpendicular to the cylinder axis. All equations are given in a container-fixed coordinate system which has its origin at the center of the undisturbed liquid free surface.

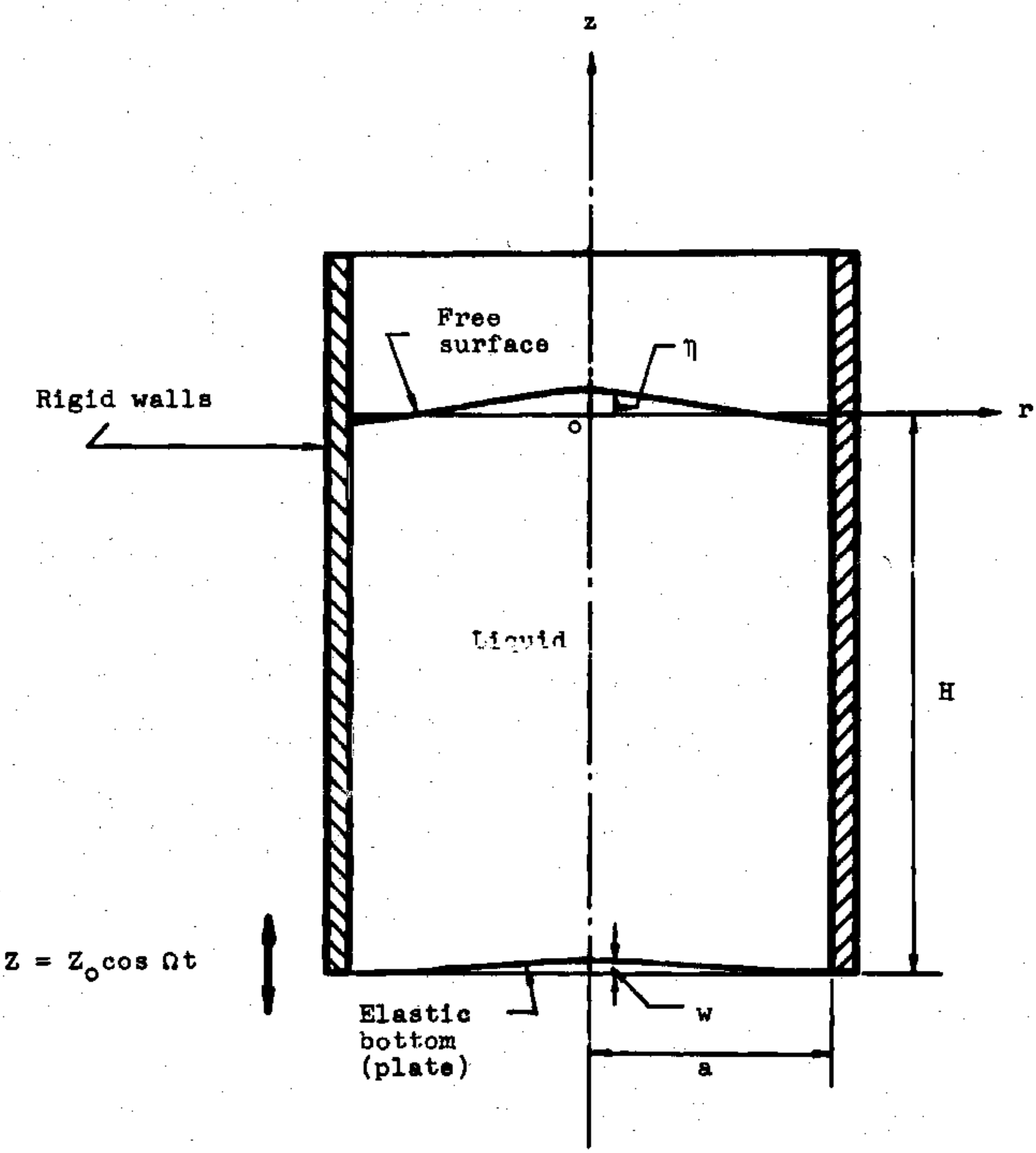


Figure 2. Tank Geometry

CHAPTER III

MATHEMATICAL FORMULATION

A circular cylindrical container of radius a as shown in Figure 2 is filled to a height H with inviscid and incompressible liquid. The container-fixed cylindrical polar coordinate system is chosen such that the origin is fixed in the center of the mean free surface, and the positive z-axis is directed upward. The deflection w of the elastic bottom is positive in the upward direction.

Since we consider the motion to be irrotational, the velocity vector of a fluid particle can be expressed by the gradient of a velocity potential Φ, i.e., the velocity components $\bar{u}$, $\bar{v}$, and $\bar{w}$ are given by

$$\bar{u} = \frac{\partial \Phi}{\partial r}, \qquad \bar{v} = \frac{1}{r}\frac{\partial \Phi}{\partial \theta}, \qquad \bar{w} = \frac{\partial \Phi}{\partial z} \tag{1}$$

The continuity equation leads to the fact that the velocity potential Φ must be a solution of the Laplace equation in the liquid domain, i.e.

$$\nabla^2 \Phi = 0 \qquad \text{for} \quad \begin{cases} 0 \leq r \leq a \\ 0 \leq \theta \leq 2\pi \\ -H + w \leq z \leq \eta \end{cases} \tag{2}$$

The kinematic conditions at the rigid tank walls and at the liquid free surface are

$$\bar{u} = \frac{\partial \Phi}{\partial r} = 0 \qquad at \qquad r = a \tag{3}$$

$$K_1(r,\theta,z,t) \equiv \frac{\partial \eta}{\partial t} + \frac{\partial \Phi}{\partial r}\frac{\partial \eta}{\partial r} + \frac{1}{r^2}\frac{\partial \Phi}{\partial \theta}\frac{\partial \eta}{\partial \theta} - \frac{\partial \Phi}{\partial z} = 0 \tag{4}$$

$$at \quad z = \eta$$

where $\eta(r,\theta,t)$ denotes the liquid free surface elevation.

At the free surface, the dynamic condition (Bernoulli's equation) yields $(p=0)$

$$D_1(r,\theta,z,t) \equiv \frac{\partial \Phi}{\partial t} + (g + \ddot{Z}(t))z + \frac{1}{2}(\nabla\Phi)^2 = 0 \qquad at\ z = \eta \tag{5}$$

where g is the gravitational acceleration, and $\ddot{Z}(t)$ is the longitudinally excited acceleration.

The kinematic condition at the tank bottom is given by

$$K_2(r,\theta,z,t) \equiv \frac{\partial w}{\partial t} + \frac{\partial \Phi}{\partial r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial \Phi}{\partial \theta}\frac{\partial w}{\partial \theta} - \frac{\partial \Phi}{\partial z} = 0 \tag{6}$$

$$at \quad z = -H + w$$

where $w(r,\theta,t)$ is the deflection of the elastic bottom, as shown in Figure 2.

The liquid pressure p on the elastic bottom can be determined from Bernoulli's equation as

$$p\Big|_{z=-H+w} = -\rho_0\left[\frac{\partial \Phi}{\partial t} + \frac{1}{2}(\nabla\Phi)^2\right]_{z=-H+w} - \rho_0(g + \ddot{Z}(t))(-H+w) \tag{7}$$

where ρ_0 denotes the density of the liquid, and H denotes the liquid height.

Since we consider the tank bottom as a thin elastic plate with clamped edges, the equation of motion of the bottom is given by

$$D\nabla^2\nabla^2 w = -\bar{\rho}\bar{h}\left[\frac{\partial^2 w}{\partial t^2} + \ddot{Z}(t) + g\right] - p\Big|_{z=-H+w} \tag{8}$$

where D is the flexural rigidity of the plate

$\bar{\rho}$ is the mass density of the plate per unit area

$\bar{h}$ is the thickness of the plate, and

p is the liquid pressure.

The boundary conditions at the elastic bottom are

$$w = 0 \qquad \text{at} \qquad r = a \tag{9}$$

$$\frac{\partial w}{\partial r} = 0 \qquad \text{at} \qquad r = a \tag{10}$$

Substituting equation (7) into equation (8) gives the equation of motion of the elastic bottom:

$$D\nabla^2\nabla^2 w = -\bar{\rho}\bar{h}\left[\frac{\partial^2 w}{\partial t^2} + \ddot{Z}(t) + g\right] + \rho_0\left[\frac{\partial \Phi}{\partial t} + \frac{1}{2}(\nabla\Phi)^2\right]_{z=-H+w} + \rho_0\left(g + \ddot{Z}(t)\right)\left(-H + w\right) \tag{11}$$

CHAPTER IV

METHOD OF SOLUTION

General Case

By using the method of separation of variables, a solution of the Laplace equation (2) satisfying the wall boundary condition (3) is obtained in the form

$$\Phi = \alpha_{oo}(t)\, z + \beta_{oo}(t) + \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} \sinh\lambda_{mn}H \left[\alpha_{mn}(t)\,\frac{\cosh\lambda_{mn}z}{\sinh\lambda_{mn}H} + \beta_{mn}(t)\,\frac{\sinh\lambda_{mn}z}{\cosh\lambda_{mn}H}\right] J_m(\lambda_{mn}r)\cos m\theta \tag{12}$$

where $\alpha_{oo}(t)$, $\beta_{oo}(t)$, $\alpha_{mn}(t)$, and $\beta_{mn}(t)$ are unknown time functions. $J_m(\lambda_{mn}r)$ is the Bessel function of the first kind and m^{th} order, and $(\lambda_{mn}a)$ are roots of the derivative of $J_m(\lambda_{mn}r)$ with respect to r, i.e.,

$$J_m'(\lambda_{mn}r)\Big|_{r=a} = 0 \quad \text{for} \quad m = 0, 1, 2, \ldots \quad n = 1, 2, 3, \ldots \tag{13}$$

The free surface of the liquid is expressed as Fourier-Bessel series

$$\eta = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} a_{mn}(t)\, J_m(\lambda_{mn}r)\cos m\theta \tag{14}$$

where the $a_{mn}(t)$ are unknown time functions.

Substituting Φ and η from equations (12) and (14) into the dynamic and kinematic conditions of the liquid free surface, equations (4) and (5), and evaluating z at the free surface $z = \eta$, yields for the dynamic condition of the liquid free surface the expression

$$
\begin{aligned}
&\dot{\alpha}_{00}(t)\eta + \dot{\beta}_{00}(t) + \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}(\sinh\lambda_{mn}H)\Big[\dot{\alpha}_{mn}(t)\frac{\cosh\lambda_{mn}\eta}{\sinh\lambda_{mn}H} \qquad (15)\\
&+\dot{\beta}_{mn}(t)\frac{\sinh\lambda_{mn}\eta}{\cosh\lambda_{mn}H}\Big]J_m(\lambda_{mn}r)\cos m\theta + (g+\ddot{Z}(t))\eta + \frac{1}{2}\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\sum_{p=0}^{\infty}\sum_{q=1}^{\infty}\\
&(\sinh\lambda_{mn}H)(\sinh\lambda_{pq}H)\Big\{\lambda_{mn}\lambda_{pq}\Big[\alpha_{mn}(t)\frac{\cosh\lambda_{mn}\eta}{\sinh\lambda_{mn}H} + \beta_{mn}(t)\frac{\sinh\lambda_{mn}\eta}{\cosh\lambda_{mn}H}\Big]\\
&\cdot\Big[\alpha_{pq}(t)\frac{\cosh\lambda_{pq}\eta}{\sinh\lambda_{pq}H} + \beta_{pq}(t)\frac{\sinh\lambda_{pq}\eta}{\cosh\lambda_{pq}H}\Big]J_m'(\lambda_{mn}r)J_p'(\lambda_{pq}r)\\
&\cdot\cos m\theta\cos p\theta + \frac{1}{r^2}mp\Big[\alpha_{mn}(t)\frac{\cosh\lambda_{mn}\eta}{\sinh\lambda_{mn}H} + \beta_{mn}(t)\frac{\sinh\lambda_{mn}\eta}{\cosh\lambda_{mn}H}\Big]\\
&\cdot\Big[\alpha_{pq}(t)\frac{\cosh\lambda_{pq}\eta}{\sinh\lambda_{pq}H} + \beta_{pq}(t)\frac{\sinh\lambda_{pq}\eta}{\cosh\lambda_{pq}H}\Big]J_m(\lambda_{mn}r)J_p(\lambda_{pq}r)\\
&\cdot\sin m\theta\sin p\theta + \lambda_{mn}\lambda_{pq}\Big[\alpha_{mn}(t)\frac{\sinh\lambda_{mn}\eta}{\sinh\lambda_{mn}H} + \beta_{mn}(t)\frac{\cosh\lambda_{mn}\eta}{\cosh\lambda_{mn}H}\Big]\\
&\cdot\Big[\alpha_{pq}(t)\frac{\sinh\lambda_{pq}\eta}{\sinh\lambda_{pq}H} + \beta_{pq}(t)\frac{\cosh\lambda_{pq}\eta}{\cosh\lambda_{pq}H}\Big]J_m(\lambda_{mn}r)J_p(\lambda_{pq}r)\cos m\theta\cos p\theta\Big\}\\
&+\frac{1}{2}\alpha_{00}^2(t) + \alpha_{00}(t)\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\lambda_{mn}\Big[\alpha_{mn}(t)\frac{\sinh\lambda_{mn}\eta}{\sinh\lambda_{mn}H} + \beta_{mn}(t)\frac{\cosh\lambda_{mn}\eta}{\cosh\lambda_{mn}H}\Big]\\
&\cdot(\sinh\lambda_{mn}H)J_m(\lambda_{mn}r)\cos m\theta = 0
\end{aligned}
$$

and for the kinematic condition the expression

$$\sum_{m=0}^{\infty}\sum_{n=1}^{\infty} \dot{a}_{mn}(t) J_m(\lambda_{mn} r)\cos m\theta + \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\sum_{p=0}^{\infty}\sum_{q=1}^{\infty} (\sinh\lambda_{mn}H) \tag{16}$$

$$\cdot\Big\{ \lambda_{mn}\lambda_{pq}\Big[\alpha_{mn}(t)\frac{\cosh\lambda_{mn}\eta}{\sinh\lambda_{mn}H} + \beta_{mn}(t)\frac{\sinh\lambda_{mn}\eta}{\cosh\lambda_{mn}H}\Big]$$

$$\cdot a_{pq}(t) J_m'(\lambda_{mn} r) J_p'(\lambda_{pq} r)\cos m\theta\cos p\theta + \frac{1}{r^2} m p$$

$$\Big[\alpha_{mn}(t)\frac{\cosh\lambda_{mn}\eta}{\sinh\lambda_{mn}H} + \beta_{mn}(t)\frac{\sinh\lambda_{mn}\eta}{\cosh\lambda_{mn}H}\Big]$$

$$\cdot a_{pq}(t) J_m(\lambda_{mn} r) J_p(\lambda_{pq} r)\sin m\theta\sin p\theta\Big\}$$

$$-\alpha_{oo}(t) - \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\lambda_{mn}(\sinh\lambda_{mn}H)$$

$$\cdot\Big[\alpha_{mn}(t)\frac{\sinh\lambda_{mn}\eta}{\sinh\lambda_{mn}H} + \beta_{mn}(t)\frac{\cosh\lambda_{mn}\eta}{\cosh\lambda_{mn}H}\Big]$$

$$\cdot J_m(\lambda_{mn} r)\cos m\theta = 0 .$$

In these expressions one can see that η occurs as the argument of hyperbolic functions. Since η itself is a double infinite series, we expand $\sinh\lambda_{mn}\eta$, $\cosh\lambda_{mn}\eta$ into series of $\lambda_{mn}\eta$. Hence

$$\sinh\lambda_{mn}\eta = \lambda_{mn}\eta + \frac{(\lambda_{mn}\eta)^3}{3!} + \frac{(\lambda_{mn}\eta)^5}{5!} + \text{-----} \tag{17}$$

$$\cosh\lambda_{mn}\eta = 1 + \frac{(\lambda_{mn}\eta)^2}{2!} + \frac{(\lambda_{mn}\eta)^4}{4!} + \text{-----} \tag{18}$$

Introducing equation (14) into equations (17) and (18), and substituting them into equations (15) and (16) yields for the dynamic condition,

$$\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\left[\dot{\alpha}_{mn}(t) + \left(g + \ddot{Z}(t)\right) a_{mn}(t)\right] J_m(\lambda_{mn} r) \cos m\theta \tag{19}$$

$$+ F\left(r, \theta, \alpha_{00}, \alpha_{01}, \beta_{01}, \beta_{11}, \alpha_{11}, ---, \dot{\alpha}_{00}, \dot{\beta}_{00}, \dot{\alpha}_{01}, \dot{\beta}_{01}, --- a_{01}, a_{11}, ---,\right) = 0$$

By considering the time functions as parameters, F can be expanded into the Fourier-Bessel series,

$$F = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} f_{mn}\left(\alpha_{00}, \alpha_{01}, \beta_{01}, \alpha_{11}, \beta_{11}, ---, \dot{\beta}_{00}, \dot{\alpha}_{01}, \dot{\beta}_{01}, ---,\right. \tag{20}$$

$$\left. a_{01}, a_{11}, ---\right) J_m(\lambda_{mn} r) \cos m\theta$$

where

$$f_{mn} = \frac{\int_0^a \int_0^{2\pi} F \cdot r \cos m\theta J_m(\lambda_{mn} r)\, d\theta\, dr}{\int_0^a \int_0^{2\pi} r \cos^2 m\theta\, J_m^2(\lambda_{mn} r)\, d\theta\, dr} \tag{21}$$

Substituting equation (20) into equation (19), results in

$$\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\left[\dot{\alpha}_{mn}(t) + \left(g + \ddot{Z}(t)\right) a_{mn}(t) + f_{mn}(t)\right] J_m(\lambda_{mn} r) \cos m\theta = 0 \tag{22}$$

Since the functions $J_m(\lambda_{mn} r)\cos m\theta$ form an orthogonal set, we obtain

$$\dot{\alpha}_{mn}(t) + (g + \ddot{Z}(t))\; a_{mn}(t) + f_{mn}(t) = 0 \tag{23}$$

for all m and n, where the $f_{mn}(t)$ are nonlinear functions of α_{mn}, β_{mn}, $\dot{\alpha}_{mn}$, $\dot{\beta}_{mn}$, and a_{mn}. Similarly the kinematic condition is expressed as

$$\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\left[\dot{a}_{mn}(t)-\lambda_{mn}(\tanh\lambda_{mn}H)\beta_{mn}(t)\right]J_m(\lambda_{mn}r)\cos m\theta \tag{24}$$

$$+G(r,\theta,\alpha_{00},\alpha_{01},\beta_{01},\alpha_{11},\beta_{11},---,\dot{a}_{01},\dot{a}_{11},---)=0$$

where the function G may be expanded into

$$G=\sum_{m=0}^{\infty}\sum_{n=1}^{\infty} g_{mn}(\alpha_{00},\alpha_{01},\beta_{01},---,\dot{a}_{01},\dot{a}_{11},---)J_m(\lambda_{mn}r)\cos m\theta \tag{25}$$

with

$$g_{mn}=\frac{\int_0^a\int_0^{2\pi} G\cdot r\cos m\theta\, J_m(\lambda_{mn}r)\,d\theta\,dr}{\int_0^a\int_0^{2\pi} r\cos^2 m\theta\, J_m^2(\lambda_{mn}r)\,d\theta\,dr} \tag{26}$$

Thus we obtain

$$\dot{a}_{mn}(t)-\lambda_{mn}(\tanh\lambda_{mn}H)\,\beta_{mn}(t)+g_{mn}(t)=0 \tag{27}$$

for all m and n.

It is of interest to note that in the case of the rigid tank bottom, equation (6) yields $\frac{\partial\Phi}{\partial z}=0$ at $z=-H$ and then from the velocity potential (12) one obtains $\alpha_{mn}(t)=\beta_{mn}(t)$. In that case, two sets of unknown time functions, $\alpha_{mn}(t)$ and $a_{mn}(t)$, have to be determined from equations (23) and (27). These represent two sets of nonlinear ordinary differential equations for $\alpha_{mn}(t)$ and $a_{mn}(t)$. In the case of an elastic bottom, however, there are three sets of unknown time functions, $\alpha_{mn}(t)$, $\beta_{mn}(t)$, and $a_{mn}(t)$, that have to be determined. In order to solve this problem completely, it is obvious that one additional relation (or one set of equations) has to be derived from the kinematic condition of the bottom (equation (6)) with the deflection w as obtained from the equation of motion of the elastic bottom. Then three equations for the unknown $\alpha_{mn}(t)$, $\beta_{mn}(t)$, and $a_{mn}(t)$ may be reduced to two equations by writing $\alpha_{mn}(t)$ in terms of $\beta_{mn}(t)$ or $\beta_{mn}(t)$ in terms of $\alpha_{mn}(t)$.

Introducing the velocity potential Φ (equation (12)) into equation (11), the equation of motion of the elastic bottom yields

$$
\begin{aligned}
D\nabla^2\nabla^2 w = -\bar{\rho}\bar{h}\,\frac{\partial^2 w}{\partial t^2} - (\rho_o H + \bar{\rho}\bar{h})(g + \ddot{Z}(t)) \qquad (28) \\
+ \rho_o\left[-\dot{\alpha}_{oo}(t)H + \dot{\beta}_{oo}(t) + \frac{1}{2}\alpha_{oo}^2(t)\right] + \rho_o\sum_{m=0}^{\infty}\sum_{n=1}^{\infty} \\
(\sinh\lambda_{mn}H)\Big\{\left[\dot{\alpha}_{mn}(t)\coth\lambda_{mn}H - \dot{\beta}_{mn}(t)\tanh\lambda_{mn}H\right] \\
+ \lambda_{mn}\alpha_{oo}(t)\left[\beta_{mn}(t) - \alpha_{mn}(t)\right]\Big\} J_m(\lambda_{mn}r)\cos m\theta \\
+ \rho_o\left(\frac{1}{2}\right)\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\sum_{p=0}^{\infty}\sum_{q=1}^{\infty}(\sinh\lambda_{mn}H)(\sinh\lambda_{pq}H) \\
\Big\{\lambda_{mn}\lambda_{pq}\left[\alpha_{mn}(t)\coth\lambda_{mn}H - \beta_{mn}(t)\tanh\lambda_{mn}H\right] \\
\left[\alpha_{pq}(t)\coth\lambda_{pq}H - \beta_{pq}(t)\tanh\lambda_{pq}H\right]J_m'(\lambda_{mn}r)J_p'(\lambda_{pq}r) \\
\cdot\cos m\theta\cos p\theta + \frac{1}{r^2}mp\Big[\alpha_{mn}(t)\coth\lambda_{mn}H
\end{aligned}
$$

$$- \beta_{mn}(t) \tanh \lambda_{mn} H \Big] \Big[\alpha_{pq}(t) \coth \lambda_{pq} H$$

$$- \beta_{pq}(t) \tanh \lambda_{pq} H \Big] J_m(\lambda_{mn} r) J_p(\lambda_{pq} r) \sin m\theta \sin p\theta$$

$$+ \lambda_{mn} \lambda_{pq} \Big[\beta_{mn}(t) - \alpha_{mn}(t) \Big] \Big[\beta_{pq}(t) - \alpha_{pq}(t) \Big]$$

$$J_m(\lambda_{mn} r) J_p(\lambda_{pq} r) \cos m\theta \cos p\theta \Big\}$$

where ∇^2 is the Laplace operator $\nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$.

Since the deflection of the plate is considerably smaller than that of the liquid free surface, the kinematic condition of the elastic bottom may be linearized and written as

$$K_2(r,\theta,z,t) \equiv \frac{\partial w}{\partial t} - \frac{\partial \Phi}{\partial z} \approx 0 \quad \text{at} \quad z = -H \tag{29}$$

Differentiating equation (29) with respect to time t yields

$$\frac{\partial^2 w}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial \Phi}{\partial z}\right)\Big|_{z=-H} \approx \dot{\alpha}_{00}(t) + \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} \lambda_{mn} (\sinh \lambda_{mn} H) \tag{30}$$

$$\cdot \Big[\dot{\beta}_{mn}(t) - \dot{\alpha}_{mn}(t) \Big] J_m(\lambda_{mn} r) \cos m\theta$$

After substituting equation (30) into equation (28), the governing differential equation of the elastic bottom is given by the expression

$$\nabla^2\nabla^2 w = -\frac{(\rho_0 H + \bar{\rho}\bar{h})}{D}\left[g + \ddot{Z}(t) + \dot{\alpha}_{00}(t)\right] + \frac{\rho_0}{D}\left[\dot{\beta}_{00}(t) + \frac{1}{2}\alpha_{00}^2(t)\right] \quad (31)$$

$$+ \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}(\sinh\lambda_{mn}H)\left\{\frac{\bar{\rho}\bar{h}}{D}\lambda_{mn}(\dot{\alpha}_{mn}(t) - \dot{\beta}_{mn}(t)) + \frac{\rho_0}{D}\right.$$

$$(\dot{\alpha}_{mn}(t)\coth\lambda_{mn}H - \dot{\beta}_{mn}(t)\tanh\lambda_{mn}H) + \frac{\rho_0}{D}\lambda_{mn}\alpha_{00}(t)$$

$$\left.(\beta_{mn}(t) - \alpha_{mn}(t))\right\} J_m(\lambda_{mn}r)\cos m\theta + \frac{\rho_0}{D}\left(\frac{1}{2}\right)\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\sum_{p=0}^{\infty}\sum_{q=1}^{\infty}$$

$$(\sinh\lambda_{mn}H)(\sinh\lambda_{pq}H)\left\{\lambda_{mn}\lambda_{pq}\left[\alpha_{mn}(t)\coth\lambda_{mn}H\right.\right.$$

$$\left.-\beta_{mn}(t)\tanh\lambda_{mn}H\right]\left[\alpha_{pq}(t)\coth\lambda_{pq}H - \beta_{pq}(t)\tanh\lambda_{pq}H\right]$$

$$J_m'(\lambda_{mn}r)\,J_p'(\lambda_{pq}r)\cos m\theta\cos p\theta + \frac{1}{r^2}mp$$

$$\cdot\left[\alpha_{mn}(t)\coth\lambda_{mn}H - \beta_{mn}(t)\tanh\lambda_{mn}H\right]$$

$$\left[\alpha_{pq}(t)\coth\lambda_{pq}H - \beta_{pq}(t)\tanh\lambda_{pq}H\right]J_m(\lambda_{mn}r)J_p(\lambda_{pq}r)$$

$$\sin m\theta\sin p\theta + \lambda_{mn}\lambda_{pq}\left[\beta_{mn}(t) - \alpha_{mn}(t)\right]\left[\beta_{pq}(t) - \alpha_{pq}(t)\right]$$

$$\left.J_m(\lambda_{mn}r)J_p(\lambda_{pq}r)\cos m\theta\cos p\theta\right\}$$

which can be rewritten in the following form

$$\nabla^2\nabla^2 w = \left[-\frac{(\rho_0 H + \bar{\rho}\bar{h})}{D}\left(g + \ddot{Z}(t) + \dot{\alpha}_{00}(t)\right) + \frac{\rho_0}{D}\left(\dot{\beta}_{00}(t) + \frac{1}{2}\alpha_{00}^2(t)\right)\right] \quad (32)$$

$$+\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}(\sinh\lambda_{mn}H)\Big\{\frac{\bar{\rho}\bar{h}}{D}\lambda_{mn}\big(\dot{\alpha}_{mn}(t)-\dot{\beta}_{mn}(t)\big)+\frac{\rho_0}{D}$$

$$\cdot\big(\dot{\alpha}_{mn}(t)\coth\lambda_{mn}H-\dot{\beta}_{mn}(t)\tanh\lambda_{mn}H\big)+\frac{\rho_0}{D}\lambda_{mn}\alpha_{00}(t)$$

$$\big(\beta_{mn}(t)-\alpha_{mn}(t)\big)\Big\}J_m(\lambda_{mn}r)\cos m\theta+K(r,\theta,\alpha_{01},\beta_{01},\alpha_{11},\beta_{11},---)$$

where K is a nonlinear function. We consider the functions of time as parameters and expand K into a Fourier-Bessel series of the form

$$K=\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}k_{mn}(\alpha_{01},\beta_{01},\alpha_{11},\beta_{11},---)J_m(\lambda_{mn}r)\cos m\theta \qquad (33)$$

where

$$k_{mn}=\frac{\int_0^a\int_0^{2\pi}K\cdot r\cdot\cos m\theta\, J_m(\lambda_{mn}r)\,d\theta\,dr}{\int_0^a\int_0^{2\pi}r\cos^2 m\theta\, J_m^2(\lambda_{mn}r)\,d\theta\,dr} \qquad (34)$$

Thus equation (32) yields

$$\nabla^2\nabla^2 w=\Big[-\frac{(\rho_0H+\bar{\rho}\bar{h})}{D}\big(g+\ddot{Z}(t)+\dot{\alpha}_{00}(t)\big)+\frac{\rho_0}{D}\big(\dot{\beta}_{00}(t)+\tfrac{1}{2}\alpha_{00}^2(t)\big)\Big] \qquad (35)$$

$$+\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}(\sinh\lambda_{mn}H)\Big\{\frac{\bar{\rho}\bar{h}}{D}\lambda_{mn}\big(\dot{\alpha}_{mn}(t)-\dot{\beta}_{mn}(t)\big)+\frac{\rho_0}{D}$$

$$\cdot\big(\dot{\alpha}_{mn}(t)\coth\lambda_{mn}H-\dot{\beta}_{mn}(t)\tanh\lambda_{mn}H\big)+\frac{\rho_0}{D}\lambda_{mn}\alpha_{00}(t)$$

$$\big(\beta_{mn}(t)-\alpha_{mn}(t)\big)+k_{mn}\Big\}J_m(\lambda_{mn}r)\cos m\theta$$

In what follows we will neglect the gravitational term g in the first bracket of equation (35). This means that the deflection of the elastic bottom w will be measured from the static equilibrium position. In other words, w will be represented as the additional deflection of the elastic bottom due to dynamic effect or the excitation of the tank.

It is to be noted that the only unknown time functions in the right hand side of equation (35) are two sets of unknown time functions, $\alpha_{mn}(t)$ and $\beta_{mn}(t)$, and their derivatives. If w, the deflection of the elastic bottom, could be found from equation (35) in terms of $\alpha_{mn}(t)$ and $\beta_{mn}(t)$, then the relation between $\alpha_{mn}(t)$ and $\beta_{mn}(t)$ could be determined from the kinematic condition of the elastic bottom (equation (29)) as

$$\beta_{mn}(t) - \alpha_{mn}(t) = \delta_{mn}(t) \tag{36}$$

where $\delta_{mn}(t)$ is a dynamic forcing function including the elastic property of the tank bottom.

For practical reasons, only a finite number of terms in the equations (15) and (16) may be retained for an approximate solution. If the liquid mode under consideration grows from the mn^{th} mode (linear theory), it is reasonable to assume that α_{mn}, β_{mn}, and a_{mn} are the predominant amplitudes in the series expansions of Φ and η.

In what follows we will restrict ourselves to the investigation of the liquid axisymmetric motion. By using the same procedure, the liquid motions of anti-symmetric modes in a circular cylindrical container with the elastic bottom may be treated in a very similar fashion.

Axisymmetric Case

In the axisymmetric case we have $m = 0$, thus Φ, η, and w are independent of the angular coordinate θ. The solution of equation (2) is given by

$$\Phi = \alpha_0(t) z + \beta_0(t) + \sum_{n=1}^{\infty} (\sinh \lambda_n H) \Big[\alpha_n(t) \frac{\cosh \lambda_n z}{\sinh \lambda_n H} \tag{37}$$

$$+ \beta_n(t) \frac{\sinh \lambda_n z}{\cosh \lambda_n H} \Big] J_0(\lambda_n r)$$

where $J_0(\lambda_n r)$ is the Bessel function of first kind and zero order.

$\alpha_0(t)$, $\beta_0(t)$, $\alpha_n(t)$, and $\beta_n(t)$ are unknown time functions, and the $(\lambda_n a)$ are the roots of

$$J_1(\lambda_n r)\Big|_{r=a} = 0 \qquad n = 1, 2, 3, \ldots \tag{38}$$

We assume that the liquid free surface is of the form of

$$\eta(r,t) = \sum_{n=1}^{\infty} a_n(t) J_0(\lambda_n r) \tag{39}$$

where $a_n(t)$ are unknown time functions which have to be determined later.

We substitute Φ, η from equations (37) and (39) into the dynamic and kinematic conditions of the liquid free surface (equations (4) and (5)) and evaluate z at the liquid free surface $z = \eta$.

The dynamic condition at the liquid free surface yields

$$\dot{\alpha}_o(t)\,\eta + \dot{\beta}(t) + \sum_{n=1}^{\infty}(\sinh\lambda_n H)\left[\dot{\alpha}_n(t)\,\frac{\cosh\lambda_n\eta}{\sinh\lambda_n H} + \dot{\beta}_n(t)\frac{\sinh\lambda_n\eta}{\cosh\lambda_n H}\right] \tag{40}$$

$$\cdot J_o(\lambda_n r) + \frac{1}{2}\sum_{n=1}^{\infty}\sum_{q=1}^{\infty}\lambda_n\lambda_q(\sinh\lambda_n H)(\sinh\lambda_q H)\left[\alpha_n(t)\,\frac{\cosh\lambda_n\eta}{\sinh\lambda_n H}\right.$$

$$\left.+\beta_n(t)\,\frac{\sinh\lambda_n\eta}{\cosh\lambda_n H}\right]\left[\alpha_q(t)\,\frac{\cosh\lambda_q\eta}{\sinh\lambda_q H} + \beta_q(t)\,\frac{\sinh\lambda_q\eta}{\cosh\lambda_q H}\right]$$

$$\cdot J_1(\lambda_n r)\,J_1(\lambda_q r) + \frac{1}{2}\left\{\alpha_o^2(t) + 2\,\alpha_o(t)\sum_{n=1}^{\infty}\lambda_n(\sinh\lambda_n H)\right.$$

$$\left[\alpha_n(t)\,\frac{\sinh\lambda_n\eta}{\sinh\lambda_n H} + \beta_n(t)\,\frac{\cosh\lambda_n\eta}{\cosh\lambda_n H}\right]J_o(\lambda_n r)$$

$$+\sum_{n=1}^{\infty}\sum_{q=1}^{\infty}\lambda_n\lambda_q(\sinh\lambda_n H)(\sinh\lambda_q H)\left[\alpha_n(t)\,\frac{\sinh\lambda_n\eta}{\sinh\lambda_n H}\right.$$

$$\left.+\beta_n(t)\,\frac{\cosh\lambda_n\eta}{\cosh\lambda_n H}\right]\left[\alpha_q(t)\,\frac{\sinh\lambda_q\eta}{\sinh\lambda_q H} + \beta_q(t)\,\frac{\cosh\lambda_q\eta}{\cosh\lambda_q H}\right]$$

$$\left.J_o(\lambda_n r)\,J_o(\lambda_q r)\right\} + (g + \ddot{Z}(t))\,\eta = 0$$

and the kinematic condition at the liquid free surface is

$$\sum_{n=1}^{\infty}\dot{a}_n(t)J_o(\lambda_n r) = \alpha_o(t) + \sum_{n=1}^{\infty}\lambda_n(\sinh\lambda_n H) \tag{41}$$

$$\cdot\left[\alpha_n(t)\,\frac{\sinh\lambda_n\eta}{\sinh\lambda_n H} + \beta_n(t)\,\frac{\cosh\lambda_n\eta}{\cosh\lambda_n H}\right]J_o(\lambda_n r)$$

$$-\sum_{n=1}^{\infty}\sum_{q=1}^{\infty}\lambda_n\lambda_q(\sinh\lambda_q H)\left[\alpha_q(t)\,\frac{\cosh\lambda_q\eta}{\sinh\lambda_q H}\right.$$

$$\left.+\beta_q(t)\,\frac{\sinh\lambda_q\eta}{\cosh\lambda_q H}\right]a_n(t)\,J_1(\lambda_n r)\,J_1(\lambda_q r)$$

Expanding again in equations (40) and (41) $\sinh\lambda_n\eta$ and $\cosh\lambda_n\eta$ into power series of $\lambda_n\eta$

$$\sinh\lambda_n\eta = \lambda_n\eta + \frac{(\lambda_n\eta)^3}{3!} + \frac{(\lambda_n\eta)^5}{5!} + \cdots \tag{42}$$

$$\cosh\lambda_n\eta = 1 + \frac{(\lambda_n\eta)^2}{2!} + \frac{(\lambda_n\eta)^4}{4!} + \cdots \tag{43}$$

and substituting η from equation (39) into equations (40) and (41), one obtains the dynamic and kinematic conditions of the liquid free surface, the expression

$$\sum_{n=1}^{\infty}\left\{\dot{\alpha}_n(t) + \left(g + \ddot{Z}(t)\right)a_n(t)\right\}J_o(\lambda_n r) \tag{44}$$

$$+ \bar{F}(r, \dot{\alpha}_o, \dot{\beta}_o, \dot{\alpha}_1, \dot{\beta}_1, ---, \alpha_o, \alpha_1, \beta_1, ---, a_1, a_2, ---) = 0$$

$\bar{\bar{F}}$ may be expanded into a Fourier-Bessel series

$$\bar{F} = \sum_{n=1}^{\infty} f_n(\alpha_o, \alpha_1, \beta_1, ---, \dot{\alpha}_o, \dot{\beta}_o, \dot{\alpha}_1, \dot{\beta}_1, ---, a_1, a_2, ---)J_o(\lambda_n r) \tag{45}$$

where

$$f_n = \frac{\int_o^a r\bar{F}J_o(\lambda_n r)\,dr}{\int_o^a rJ_o^2(\lambda_n r)\,dr} \tag{46}$$

The dynamic condition yields then

$$\sum_{n=1}^{\infty} \left\{ \dot{\alpha}_n(t) + \left(g + \ddot{Z}(t) \right) a_n(t) + f_n(t) \right\} J_o(\lambda_n r) = 0 \tag{47}$$

which, due to the orthogonality of the Bessel function is satisfied by

$$\dot{\alpha}_n(t) + (g + \ddot{Z}(t))\, a_n(t) + f_n = 0 \tag{48}$$

for all n. The functions $f_n(t)$ are nonlinear functions of α_n, β_n, $\dot{\alpha}_n$, $\dot{\beta}_n$, and a_n. Similarly, one obtains for the kinematic condition (equation (41))

$$\sum_{n=1}^{\infty} \left\{ \dot{a}_n(t) - \lambda_n (\tanh \lambda_n H) \beta_n(t) \right\} J_o(\lambda_n r) \tag{49}$$

$$+ \bar{G}(r, \alpha_o, \alpha_1, \beta_1, ---, \dot{a}_1, \dot{a}_2, ---) = 0$$

which with the Fourier-Bessel series expansion

$$\bar{G} = \sum_{n=1}^{\infty} g_n(\alpha_o, \alpha_1, \beta_1, ---, \dot{a}_1, \dot{a}_2, ---) J_o(\lambda_n r) \tag{50}$$

where

$$g_n = \frac{\int_o^a r \cdot \bar{G}\, J_o(\lambda_n r)\, dr}{\int_o^a r\, J_o^2(\lambda_n r)\, dr} \tag{51}$$

yields

$$\dot{a}_n(t) - \lambda_n(\tanh \lambda_n H)\, \beta_n(t) + g_n(t) = 0 \tag{52}$$

To obtain the third set of equations for determination of the unknown functions $\alpha_n(t)$, $\beta_n(t)$, and $a_n(t)$, we have to solve the equation for the axisymmetric motion of the elastic bottom.

$$D\nabla^2\nabla^2 w = -\bar{\rho}\bar{h}\frac{\partial^2 w}{\partial t^2} - (\rho_o H + \bar{\rho}\bar{h})(\ddot{Z}(t)) + \rho_o\Big\{-\dot{\alpha}_o(t)H + \dot{\beta}_o(t) \tag{53}$$

$$+\sum_{n=1}^{\infty}(\sinh\lambda_n H)\left[\dot{\alpha}_n(t)\coth\lambda_n H - \dot{\beta}_n(t)\tanh\lambda_n H\right]J_o(\lambda_n r)\Big\}$$

$$+\rho_o\left(\tfrac{1}{2}\right)\Big\{\alpha_o^2(t) + 2\alpha_o(t)\sum_{n=1}^{\infty}\lambda_n(\sinh\lambda_n H)(\beta_n(t)-\alpha_n(t))J_o(\lambda_n r)$$

$$+\sum_{n=1}^{\infty}\sum_{g=1}^{\infty}\lambda_n\lambda_g(\sinh\lambda_n H)(\sinh\lambda_g H)(\beta_n(t)-\alpha_n(t))(\beta_g(t)-$$

$$\alpha_g(t))J_o(\lambda_n r)J_o(\lambda_g r) + \sum_{n=1}^{\infty}\sum_{g=1}^{\infty}\lambda_n\lambda_g(\sinh\lambda_n H)(\sinh\lambda_g H)$$

$$\cdot(\alpha_n(t)\coth\lambda_n H - \beta_n(t)\tanh\lambda_n H)(\alpha_g\coth\lambda_g H - \beta_g\tanh\lambda_g H)J_1(\lambda_n r)J_1(\lambda_g r)\Big\}$$

here $\nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}$ is the Laplace operator for the axisymmetric case.

By considering the kinematic condition (29), as in the previous case, the kinematic condition is differentiated with respect to time in order to eliminate the value $\frac{\partial^2 w}{\partial t^2}$ in equation (53). Now the right hand side of this equation depends only on the velocity potential and is given by the expression

$$\nabla^2\nabla^2 w = -\frac{(\rho_o H + \bar{\rho}\bar{h})}{D}\left[\ddot{Z}(t) + \dot{\alpha}_o(t)\right] + \frac{\rho_o}{D}\left(\dot{\beta}_o(t) + \frac{1}{2}\alpha_o^2(t)\right) \tag{54}$$

$$+\sum_{n=1}^{\infty}(\sinh\lambda_n H)\left\{\frac{\bar{\rho}\bar{h}}{D}\lambda_n\left(\dot{\alpha}_n(t)-\dot{\beta}_n(t)\right)+\frac{\rho_o}{D}\left(\dot{\alpha}_n(t)\coth\lambda_n H\right.\right.$$

$$\left.\left.-\dot{\beta}_n(t)\tanh\lambda_n H\right)+\frac{\rho_o}{D}\lambda_n\alpha_o(t)\left(\beta_n(t)-\alpha_n(t)\right)\right\}J_o(\lambda_n r)$$

$$+\bar{K}(r,\alpha_1,\beta_1,\alpha_2,\beta_2,---)$$

where $\bar{K}$ is a nonlinear function given by

$$\bar{K}=\frac{\rho_o}{D}\left(\frac{1}{2}\right)\sum_{n=1}^{\infty}\sum_{g=1}^{\infty}(\sinh\lambda_n H)(\sinh\lambda_g H)\left\{\lambda_n\lambda_g\left[\alpha_n(t)\coth\lambda_n H\right.\right. \tag{55}$$

$$\left.-\beta_n(t)\tanh\lambda_n H\right]\left[\alpha_g(t)\coth\lambda_g H-\beta_g(t)\tanh\lambda_g H\right]J_1(\lambda_n r)J_1(\lambda_g r)$$

$$\left.+\lambda_n\lambda_g\left[\beta_n(t)-\alpha_n(t)\right]\left[\beta_g(t)-\alpha_g(t)\right]J_o(\lambda_n r)J_o(\lambda_g r)\right\}$$

Expanding $\bar{K}$ into a Fourier-Bessel series

$$\bar{K}=\sum_{n=1}^{\infty}k_n(\alpha_1,\beta_1,\alpha_2,\beta_2,---)J_o(\lambda_n r) \tag{56}$$

where

$$k_n=\frac{\int_o^a r\,\bar{K}\,J_o(\lambda_n r)\,dr}{\int_o^a r\,J_o^2(\lambda_n r)\,dr} \tag{57}$$

and substituting it into equation (54), yields for the equation of motion of the elastic plate:

$$\nabla^2\nabla^2 w=\left[-\frac{(\rho_o H+\bar{\rho}\bar{h})}{D}\left(\ddot{Z}(t)+\dot{\alpha}_o(t)\right)+\frac{\rho_o}{D}\left(\dot{\beta}(t)+\frac{1}{2}\alpha_o^2(t)\right)\right] \tag{58}$$

$$+\sum_{n=1}^{\infty}(\sinh\lambda_n H)\left\{\frac{\bar{\rho}\bar{k}}{D}\lambda_n\left(\dot{\alpha}_n(t)-\dot{\beta}_n(t)\right)+\frac{\rho_o}{D}\left(\dot{\alpha}_n(t)\coth\lambda_n H\right.\right.$$

$$\left.\left.-\dot{\beta}_n(t)\tanh\lambda_n H\right)+\frac{\rho_o}{D}\lambda_n\alpha_o(t)\left(\beta_n(t)-\alpha_n(t)\right)+k_n\right\}J_o(\lambda_n r)$$

It is to be noted again that the only unknown functions on the right hand side of equation (58) are $\alpha_n(t)$ and $\beta_n(t)$ as well as their derivatives. A solution of equation (58), $w(r,t)$, is then in terms of $\alpha_n(t)$ and $\beta_n(t)$ and shall yield with the kinematic condition (29) a relation between $\alpha_n(t)$ and $\beta_n(t)$. For an approximate solution only a finite number of terms are retained. In the following analysis, both uncoupled fundamental axisymmetric mode and those coupled with the second axisymmetric mode will be considered.

The orders of α_n, β_n, and a_n are assumed to be

$$\alpha_1 = O(A), \quad \alpha_2 = O(A^{(2)}), \quad \alpha_n = O(A^{(n)}) \tag{59}$$

$$\beta_1 = O(A), \quad \beta_2 = O(A^{(2)}), \quad \beta_n = O(A^{(n)})$$

$$a_1 = O(A), \quad a_2 = O(A^{(2)}), \quad a_n = O(A^{(n)})$$

where A is the amplitude corresponding to the first axisymmetric mode of the free oscillation (linear).

From previous studies of rectangular and cylindrical containers, it appears that terms at least up to <u>third order</u> must be retained in order to determine the finite, steady state amplitude of the liquid response.

According to the assumption of orders which were made in equation

(59) and by using the power series expansion of $\sinh\lambda_n\eta$ and $\cosh\lambda_n\eta$, the dynamic and kinematic conditions of the liquid free surface, after truncating up to third order, can be written explicitely as follows.

The dynamic condition is

$$\Big\{\big[\dot{\alpha}_1(t) + \big(g + \ddot{Z}(t)\big)\,a_1(t)\big]J_0(\lambda_1 r) + \big[\dot{\alpha}_2(t) + \big(g + \ddot{Z}(t)\big)\,a_2(t)\big]J_0(\lambda_2 r)\Big\} + \bar{F} = 0 \tag{60}$$

where

$$\begin{aligned}\bar{F} \equiv\ & \dot{\beta}_0(t) + \frac{1}{2}\alpha_0^2(t) + \dot{\alpha}_0(t)a_1(t)J_0(\lambda_1 r) + \dot{\alpha}_0(t)a_2(t)J_0(\lambda_2 r) + \frac{1}{2}\lambda_1^2 \\ & \cdot a_1^2(t)\dot{\alpha}_1(t)J_0^3(\lambda_1 r) + \lambda_1(\tanh\lambda_1 H)a_1(t)\dot{\beta}_1(t)J_0^2(\lambda_1 r) + \big[\lambda_1(\tanh\lambda_1 H) \\ & \cdot a_2(t)\dot{\beta}_1(t) + \lambda_2(\tanh\lambda_2 H)a_1(t)\dot{\beta}_2(t)\big]J_0(\lambda_1 r)J_0(\lambda_2 r) + \frac{1}{2}\lambda_1^2\alpha_1^2(t) \\ & \cdot J_1^2(\lambda_1 r) + \lambda_1\lambda_2\alpha_1(t)\alpha_2(t)J_1(\lambda_1 r)J_2(\lambda_2 r) + \lambda_1^3(\tanh\lambda_1 H)\alpha_1(t)\beta_1(t)a_1(t) \\ & \cdot J_0(\lambda_1 r)J_1^2(\lambda_1 r) + \lambda_1(\tanh\lambda_1 H)\alpha_0(t)\beta_1(t)J_0(\lambda_1 r) + \lambda_2(\tanh\lambda_2 H)\alpha_0(t) \\ & \cdot \beta_2(t)J_0(\lambda_2 r) + \lambda_1^2\alpha_0(t)\alpha_1(t)a_1(t)J_0^2(\lambda_1 r) + \frac{1}{2}\lambda_1^2(\tanh\lambda_1 H)^2\beta_1^2(t) \\ & \cdot J_0^2(\lambda_1 r) + \lambda_1\lambda_2(\tanh\lambda_1 H)(\tanh\lambda_2 H)\beta_1(t)\beta_2(t)J_0(\lambda_1 r)J_0(\lambda_2 r)\end{aligned} \tag{61}$$

$$+ \lambda_1^3 (\tanh \lambda_1 H) \alpha_1(t) \beta_1(t) a_1(t) J_0^3(\lambda_1 r)$$

and the kinematic condition is

$$\Big\{ \big[\dot{a}_1(t) - \lambda_1 (\tanh \lambda_1 H) \beta_1(t) \big] J_0(\lambda_1 r) + \big[\dot{a}_2(t) - \lambda_2 (\tanh \lambda_2 H) \beta_2(t) \big] J_0(\lambda_2 r) \Big\} + \bar{G} = 0 \tag{62}$$

where

$$\begin{aligned} \bar{G} \equiv & -\lambda_1^2 \alpha_1(t) a_1(t) J_0^2(\lambda_1 r) - \big(\lambda_1^2 \alpha_1(t) a_2(t) + \lambda_2^2 \alpha_2(t) a_1(t) \big) \\ & \cdot J_0(\lambda_1 r) J_0(\lambda_2 r) - \frac{1}{2} \lambda_1^3 (\tanh \lambda_1 H) \beta_1(t) a_1^2(t) J_0^3(\lambda_1 r) + \lambda_1^2 a_1(t) \\ & \cdot \alpha_1(t) J_1^2(\lambda_1 r) + \big(\lambda_1 \lambda_2 a_1(t) \alpha_2(t) + \lambda_1 \lambda_2 a_2(t) \alpha_1(t) \big) J_1(\lambda_1 r) J_1(\lambda_2 r) \\ & + \lambda_1^3 (\tanh \lambda_1 H) a_1^2(t) \beta_1(t) J_0(\lambda_1 r) J_1^2(\lambda_1 r) \end{aligned} \tag{63}$$

In the above equations (60) and (62), the nonlinear functions $\bar{F}$ and $\bar{G}$ have to be expanded into Fourier-Bessel series yielding the expression of the form equations (45) and (50), of which a third order approximation may be obtained easily.

The equation of motion of the elastic bottom is then given by

$$\nabla^2 \nabla^2 w = - \frac{(\rho_0 H + \bar{\rho} \bar{h})}{D} \big[\ddot{Z}(t) + \dot{\alpha}_0(t) \big] + \frac{\rho_0}{D} \Big(\dot{\beta}_0(t) + \frac{1}{2} \alpha_0^2(t) \Big) \tag{64}$$

$$+\Big\{\Big[\frac{\bar{\rho}\bar{h}}{D}\lambda_1\left(\dot{\alpha}_1(t)-\dot{\beta}_1(t)\right)+\frac{\rho_0}{D}\left(\dot{\alpha}_1(t)\coth\lambda_1 H-\dot{\beta}_1(t)\tanh\lambda_1 H\right)$$

$$+\frac{\rho_0}{D}\lambda_1\alpha_0(t)\left(\beta_1(t)-\alpha_1(t)\right)\Big](\sinh\lambda_1 H)J_0(\lambda_1 r)+\Big[\frac{\bar{\rho}\bar{h}}{D}\lambda_2$$

$$\cdot\left(\dot{\alpha}_2(t)-\dot{\beta}_2(t)\right)+\frac{\rho_0}{D}\left(\dot{\alpha}_2(t)\coth\lambda_2 H-\dot{\beta}_2(t)\tanh\lambda_2 H\right)$$

$$+\frac{\rho_0}{D}\lambda_2\alpha_0(t)\left(\beta_2(t)-\alpha_2(t)\right)\Big](\sinh\lambda_2 H)J_0(\lambda_2 r)\Big\}+\bar{K}$$

where

$$\bar{K}=\frac{\rho_0}{D}\Big\{\frac{1}{2}\lambda_1^2(\sinh\lambda_1 H)^2\left(\beta_1(t)-\alpha_1(t)\right)^2 J_0^2(\lambda_1 r)+\lambda_1\lambda_2(\sinh\lambda_1 H)(\sinh\lambda_2 H) \quad (65)$$

$$\cdot\left(\beta_1(t)-\alpha_1(t)\right)\left(\beta_2(t)-\alpha_2(t)\right)J_0(\lambda_1 r)J_0(\lambda_2 r)+\frac{1}{2}\lambda_1^2(\sinh\lambda_1 H)^2\left(\alpha_1(t)\coth\lambda_1 H\right.$$

$$\left.-\beta_1(t)\tanh\lambda_1 H\right)^2 J_1^2(\lambda_1 r)+\lambda_1\lambda_2(\sinh\lambda_1 H)(\sinh\lambda_2 H)\left(\alpha_1(t)\coth\lambda_1 H\right.$$

$$\left.-\beta_1(t)\tanh\lambda_1 H\right)\left(\alpha_2(t)\coth\lambda_2 H-\beta_2(t)\tanh\lambda_2 H\right)J_1(\lambda_1 r)J_1(\lambda_2 r)$$

The kinematic condition at the elastic bottom is

$$\frac{\partial w}{\partial t}=\alpha_0(t)+\lambda_1(\sinh\lambda_1 H)\left(\beta_1(t)-\alpha_1(t)\right)J_0(\lambda_1 r) \quad (66)$$

$$+\lambda_2(\sinh\lambda_2 H)\left(\beta_2(t)-\alpha_2(t)\right)J_0(\lambda_2 r)$$

To solve equation (64), we expand $\bar{K}$ into Fourier-Bessel series, thus giving for the equation of the plate

$$\nabla^2\nabla^2 w = \frac{1}{D}\left\{ q_0(t) + q_1(t)J_0(\lambda_1 r) + q_2(t)J_0(\lambda_2 r) \right\} \tag{67}$$

where

$$\begin{aligned} q_0(t) \equiv & -(\rho_0 H + \bar{\rho}\bar{h})[\ddot{Z}(t) + \dot{\alpha}_0(t)] + \rho_0(\dot{\beta}_0(t) + \tfrac{1}{2}\alpha_0^2(t)) \\ & + \rho_0(0.1622151)\tfrac{1}{2}\lambda_1^2(\sinh\lambda_1 H)^2(\beta_1(t) - \alpha_1(t))^2 + \rho_0(0.1622594) \\ & \cdot\tfrac{1}{2}\lambda_1^2(\sinh\lambda_1 H)^2(\alpha_1(t)\coth\lambda_1 H - \beta_1\tanh\lambda_1 H)^2 \end{aligned} \tag{68}$$

$$\begin{aligned} q_1(t) \equiv & \bar{\rho}\bar{h}\lambda_1(\sinh\lambda_1 H)(\dot{\alpha}_1(t) - \dot{\beta}_1(t)) + \rho_0(\sinh\lambda_1 H)(\dot{\alpha}_1(t)\coth\lambda_1 H \\ & -\dot{\beta}_1(t)\tanh\lambda_1 H) + \rho_0\lambda_1(\sinh\lambda_1 H)\alpha_0(t)(\beta_1(t) - \alpha_1(t)) + \rho_0(0.3522803) \\ & \tfrac{1}{2}\lambda_1^2(\sinh\lambda_1 H)^2(\beta_1(t) - \alpha_1(t))^2 + \rho_0(0.2661404)\lambda_1\lambda_2(\sinh\lambda_1 H) \\ & \cdot(\sinh\lambda_2 H)(\beta_1(t) - \alpha_1(t))(\beta_2(t) - \alpha_2(t)) + \rho_0(0.1761056)\tfrac{1}{2}\lambda_1^2 \\ & (\sinh\lambda_1 H)^2(\alpha_1(t)\coth\lambda_1 H - \beta_1(t)\tanh\lambda_1 H)^2 + \rho_0(0.2436875) \\ & \lambda_1\lambda_2(\sinh\lambda_1 H)(\sinh\lambda_2 H)(\alpha_1(t)\coth\lambda_1 H - \beta_1(t)\tanh\lambda_1 H) \\ & \cdot(\alpha_2(t)\coth\lambda_2 H - \beta_2(t)\tanh\lambda_2 H) \end{aligned} \tag{69}$$

$$q_2(t) = \bar{\rho}\bar{h}\lambda_2(\sinh\lambda_2 H)(\dot{\alpha}_2(t) - \dot{\beta}_2(t)) + \rho_o(\sinh\lambda_2 H)(\dot{\alpha}_2(t)\coth\lambda_2 H \tag{70}$$

$$- \dot{\beta}_2(t)\tanh\lambda_2 H) + \rho_o\lambda_2(\sinh\lambda_2 H)\alpha_o(t)(\beta_2(t) - \alpha_2(t)) + \rho_o(0.4793192)$$

$$\cdot\frac{1}{2}\lambda_1^2(\sinh\lambda_1 H)^2(\beta_1(t) - \alpha_1(t))^2 + \rho_o(0.3019649)\lambda_1\lambda_2(\sinh\lambda_1 H)$$

$$(\sinh\lambda_2 H)(\beta_1(t) - \alpha_1(t))(\beta_2(t) - \alpha_2(t)) - \rho_o(0.3241766)\frac{1}{2}\lambda_1^2(\sinh\lambda_1 H)^2$$

$$(\alpha_1(t)\coth\lambda_1 H - \beta_1(t)\tanh\lambda_1 H)^2 + \rho_o(0.0824586)\lambda_1\lambda_2(\sinh\lambda_1 H)(\sinh\lambda_2 H)$$

$$\cdot(\alpha_1(t)\coth\lambda_1 H - \beta_1(t)\tanh\lambda_1 H)(\alpha_2(t)\coth\lambda_2 H - \beta_2(t)\tanh\lambda_2 H)$$

A solution of equation (67), $w(r,t)$, which is bounded in the region $0 \leqq r \leqq a$ and satisfying boundary conditions $w\big|_{r=a} = 0$ and $\frac{\partial w}{\partial r}\big|_{r=a} = 0$ is given by

$$w(r,t) = \frac{1}{D}\left\{-\frac{q_o(t)}{64}(a^2 - r^2)^2 + \frac{q_1(t)}{\lambda_1^4}(J_o(\lambda_1 r) - J_o(\lambda_1 a)) + \frac{q_2(t)}{\lambda_2^4}(J_o(\lambda_2 r) - J_o(\lambda_2 a))\right\} \tag{71}$$

With this result and equation (37), the kinematic condition yields

$$\frac{\dot{q}_o(t)}{64}(a^4 - 2a^2r^2 + r^4) + \frac{\dot{q}_1(t)}{\lambda_1^4}(J_o(\lambda_1 r) - J_o(\lambda_1 a)) + \frac{\dot{q}_2(t)}{\lambda_2^4}(J_o(\lambda_2 r) - J_o(\lambda_2 a)) \tag{72}$$

$$= D\alpha_o(t) + D\lambda_1(\beta_1(t) - \alpha_1(t))J_o(\lambda_1 r) + D\lambda_2(\beta_2(t) - \alpha_2(t))J_o(\lambda_2 r)$$

Expanding the terms r^2 and r^4 on the left hand side of equation (72) into Fourier-Bessel series of $J_0(\lambda_n r)$ and comparing the coefficients of $J_0(\lambda_n r)$ on both sides yields a set of equations between $\alpha_n(t)$ and $\beta_n(t)$. They are given by

$$\alpha_0(t) = \frac{a^4}{D}\left[(0.005208)\dot{q}_0(t) + (0.001868)\dot{q}_1(t) - (0.000124)\dot{q}_2(t)\right] \quad (73)$$

$$\beta_1(t) - \alpha_1(t) = \delta_1(t) = \frac{a^5}{D\sinh\lambda_1 H}\left[(0.001906)\dot{q}_0(t) + (0.001211)\dot{q}_1(t)\right] \quad (74)$$

$$\beta_2(t) - \alpha_2(t) = \delta_2(t) = \frac{a^5}{D\sinh\lambda_2 H}\left[(0.000195)\dot{q}_0(t) + (0.000059)\dot{q}_2(t)\right] \quad (75)$$

where the values of $(\lambda_1 a) = 3.8317060$, $(\lambda_2 a) = 7.0155867$, $J_0(\lambda_1 a) = -0.4027594$, and $J_0(\lambda_2 a) = 0.3001158$ have been used.

For a rigid container bottom $(D \to \infty)$ one obtains the values $\alpha_0(t) = 0$, $\beta_1(t) = \alpha_1(t)$, $\beta_2(t) = \alpha_2(t)$ as expected. From the above we find that the difference of $\alpha_n(t)$ from $\beta_n(t)$ is decreasing as n increased. For example, when $n = 3$, the value $\beta_3(t) - \alpha_3(t) \approx 0$ indicating that for higher modes the effect of an elastic bottom does not appreciably differ from that of a rigid bottom.

The procedure of solving this problem (for axisymmetric case) can be summarized as follows:

(1) The assumed form of Φ, η from equations (37) and (39) is substituted into the dynamic and kinematic conditions of the liquid free surface (equations (40) and (41)).

(2) In equations (40) and (41) $\sinh\lambda_n\eta$ and $\cosh\lambda_n\eta$ are expanded into a power series of $\lambda_n\eta$.

(3) Using the assumed orders of α_n, β_n, and a_n in equation (59), terms in equations (40) and (41) up to third order are truncated.

(4) Those terms of $\overline{F}$, $\overline{G}$, and $\overline{K}$ in equations (61), (63), and (65) are expanded into Fourier-Bessel series of $J_0(\lambda_n r)$, $n = 1, 2, 3, \ldots$, respectively. This is valid since the set of functions $J_0(\lambda_n r)$ forms an orthogonal complete set in $0 \leq r \leq a$.[20]

(5) In equation (60) all terms of $J_0(\lambda_n r)$ for each n are collected and let the coefficient corresponding to each $J_0(\lambda_n r)$ be zero, from which we obtain a set of simultaneous nonlinear ordinary differential equations with undetermined time functions of $\alpha_n(t)$, $\beta_n(t)$, and $a_n(t)$ as shown in equation (48). The same procedure is performed in equation (62) to get a set of equations as shown in equation (52).

(6) According to the assumption of orders which were made in equation (59), those terms on the right hand side of equation (58) are truncated up to the third order and try to find a solution from equation (64) in terms of $\alpha_n(t)$ and $\beta_n(t)$, which satisfies equation (64) as well as the boundary conditions $w\Big|_{r=a} = 0$ and $\frac{\partial w}{\partial r}\Big|_{r=a} = 0$.

(7) The kinematic condition of the elastic bottom is used to find the relation between $\beta_n(t)$ and $\alpha_n(t)$ as $\beta_n(t) - \alpha_n(t) = \delta_n(t)$ where $\delta_n(t)$ is some external forcing function involving the elastic property of the bottom (plate rigidity).

(8) The relation of $\alpha_n(t)$ and $\beta_n(t)$ is substituted into the kinematic and dynamic conditions of the liquid free surface (equations (60) and (62)), which after their elimination yield coupled equation for $a_n(t)$.

One nonlinear ordinary differential equation of $a_1(t)$ is obtained for uncoupled axisymmetric liquid motion. Two coupled nonlinear ordinary differential equations of $a_1(t)$ and $a_2(t)$ are obtained for coupled axisymmetric liquid motion.

(9) After $a_n(t)$, $\alpha_n(t)$, and $\beta_n(t)$ are determined, the velocity potential Φ, the elevation of the liquid free surface η, the pressure distribution p, and the liquid force and moment can be obtained without any further difficulties.

In the following we restrict the longitudinal excitation function to be

$$Z(t) = Z_0 \cos\Omega t \tag{76}$$

where Z_0 is the excitation amplitude, and Ω is the forcing frequency. The range of Z_0 is restricted to a small value only.

Experiments have shown that, when a liquid in a circular cylindrical container is subjected to longitudinal excitations, the free surface of the liquid may exhibit a plane surface or it may have a periodic motion of finite amplitude depending on the excitation parameters, Z_0 and Ω as well as the elastic effect of the bottom. Furthermore, the liquid motion might be a subharmonic, harmonic, or superharmonic and can be composed of a variety of modes depending on the excitation parameters. In the following, two major cases will be considered. First, we assume that the fluid motion exhibits no coupling of modes, meaning that all other modes may be neglected. In the second case, we assume that coupling is allowed. From the dynamics point of view, the first symmetric mode and first antisymmetric mode are the most important to the stability of the space vehicles. Especially the first symmetric mode is of some detailed interest for the interaction problem of pump-, combustion-, feedline-, propellant-, and structural dynamics.

Uncoupled Motion (First Symmetric Mode)

As a first approximation we assume that there is no coupling of modes. Then the dynamic condition of the liquid free surface from equation (48) with n = 1

$$\{\dot{\alpha}_1(t) + (g + \ddot{Z}(t))\, a_1(t)\} + f_1 = 0 \tag{77}$$

where

$$f_1(t) \equiv \dot{\alpha}_0(t)\, a_1(t) + \lambda_1 (\tanh \lambda_1 H) \alpha_0(t) \beta_1(t) + (0.3522803) \tag{78}$$

$$\cdot \left[\lambda_1 (\tanh \lambda_1 H)\, a_1(t) \dot{\beta}_1(t) + \lambda_1^2 \alpha_0(t) \alpha_1(t)\, a_1(t) + \frac{1}{2} \lambda_1^2 (\tanh \lambda_1 H)^2 \beta_1^2(t) \right]$$

$$+ (0.1761056) \frac{1}{2} \lambda_1^2 \alpha_1^2(t) + (0.1379748) \lambda_1^3 (\tanh \lambda_1 H)\, \alpha_1(t) \beta_1(t)\, a_1(t)$$

$$+ (0.4138998) \left[\frac{1}{2} \lambda_1^2 a_1^2(t) \dot{\alpha}_1(t) + \lambda_1^3 (\tanh \lambda_1 H)\, \alpha_1(t) \beta_1(t)\, a_1(t) \right]$$

The kinematic condition of the liquid free surface follows from equation (52) and for n = 1 is given by

$$\{\dot{a}_1(t) - \lambda_1 (\tanh \lambda_1 H)\, \beta_1(t)\} + g_1 = 0 \tag{79}$$

where

$$g_1 \equiv -(0.3522803)\lambda_1^2 \alpha_1(t)\, a_1(t) + (0.1761056)\lambda_1^2 a_1(t)\alpha_1(t) \quad (80)$$

$$-(0.4138998)\frac{1}{2}\lambda_1^3 (\tanh\lambda_1 H)\beta_1(t)\, a_1^2(t)$$

$$+(0.1379748)\lambda_1^3 (\tanh\lambda_1 H)\, a_1^2(t)\beta_1(t)$$

The above equations (77) and (79) together with equation (74) constitute the system of equations for the determination of the $\alpha_1(t)$, $\beta_1(t)$, and $a_1(t)$.

Combining these equations and neglecting terms of higher than third order yields an equation in terms $A_1(t) = \frac{a_1(t)}{a}$ of the form

$$\ddot{A}_1(t) + \omega_1^2\left(1 - \frac{Z_0}{g}\Omega^2\cos\Omega t\right)\left[A_1(t) + Q_1 A_1^2(t) + Q_2 A_1^3(t)\right] \quad (81)$$

$$+ Q_3 \dot{A}_1^2(t) + Q_4 A_1(t)\ddot{A}_1(t) + Q_5 A_1(t)\dot{A}_1^2(t) + Q_6 A_1^2(t)\ddot{A}_1(t)$$

$$-(P_1 Q_7 Z_0 \Omega^4)\cos\Omega t\, A_1(t) - (P_1 Q_8 Z_0 \Omega^3)\sin\Omega t\, \dot{A}_1(t)$$

$$= -(a \times P_1 Z_0 \Omega^4)\cos\Omega t$$

where the excitation function has been taken as

$$Z = Z_0\cos\Omega t \quad \text{and} \quad X = \lambda_1\tanh(\lambda_1 H)$$

The coefficients of the nonlinear terms Q_1 through Q_8 are listed in

Appendix B. They all depend upon the liquid height H and the radius a of the tank. The expression

$$\omega_1^2 = \lambda_1 \ g \ \tanh(\lambda_1 H) \tag{82}$$

is the square of the linearized natural frequency of the liquid for the fundamental axisymmetric mode in a rigid circular cylindrical container. Furthermore, P_1 is given by

$$P_1 = \frac{(0.001906)a^3(\rho_0 H + \bar{\rho}\bar{h})}{D(\sinh\lambda_1 H)} \tag{83}$$

and depends on the liquid density, the liquid height, the density of the plate, the thickness of the plate, and the stiffness of the bottom.

It is of interest to note that the nonhomogeneous term appearing on the right hand side of equation (81) is due to the elastic effect of the bottom. For the limiting case of the plate rigidity, $D \to \infty$, approaching infinity, the value of $P_1 \to 0$, exhibiting that the nonhomogeneous term on the right hand side of equation (81) is becoming zero. Equation (81) then coincides with that for the case of a container with a rigid bottom. However, in the case of an elastic bottom, the nonhomogeneous term with increasing radius becomes more significant, even though the excitation amplitude is restricted to a range of small values.

If equation (81) is linearized, it yields

$$\ddot{A}_1(t) + \omega_1^2\left(1 - \frac{Z_0}{g}\,\Omega^2 \cos\Omega t\right)A_1(t) = -(a)(P_1 Z_0 \Omega^4)\cos\Omega t \tag{84}$$

which represents a nonhomogeneous Mathieu equation.[21] To investigate the solution of equation (81), the nonlinear uncoupled one-half subharmonic and harmonic liquid responses will be analyzed.

One-half Subharmonic Response. A solution of equation (81) may be written as

$$A_1(t) = \xi \sin \tfrac{1}{2}\Omega t \tag{85}$$

After substituting this expression into equation (81), the response function yields

$$\xi\Big\{\Big[1-\bar{r}^2+2\Big(\frac{Z_o}{g}\Big)\omega_1^2\bar{r}^2+(8Q_7-4Q_8)P_1 Z_o\omega_1^2\bar{r}^4\Big] \tag{86}$$

$$+\xi^2\Big[\frac{3}{4}Q_2+2Q_2\Big(\frac{Z_o}{g}\Big)\omega_1^2\bar{r}^2+\Big(\frac{1}{4}Q_5-\frac{3}{4}Q_6\Big)\bar{r}^2\Big]\Big\}=0$$

where

$$\bar{r} = \left(\frac{\frac{\Omega}{2}}{\omega_1}\right) \tag{87}$$

From this we conclude that either

$$\xi = 0$$

or

$$\xi^2 = -\frac{\Big[1-\bar{r}^2+2\Big(\frac{Z_o}{g}\Big)\omega_1^2\bar{r}^2+(8Q_7-4Q_8)P_1 Z_o\omega_1^2\bar{r}^4\Big]}{\Big[\frac{3}{4}Q_2+2Q_2\Big(\frac{Z_o}{g}\Big)\omega_1^2\bar{r}^2+\Big(\frac{1}{4}Q_5-\frac{3}{4}Q_6\Big)\bar{r}^2\Big]} \tag{88}$$

Another approximate solution of equation (81) can be written as

$$A_1(t) = \bar{\xi} \cos \tfrac{1}{2}\Omega t \tag{89}$$

We substitute equation (89) into equation (81), then the response function yields

$$\bar{\xi}\left\{\left[1-\bar{r}^2-2\left(\frac{Z_o}{g}\right)\omega_1^2\bar{r}^2-(8Q_7-4Q_8)P_1 Z_o\omega_1^2\bar{r}^4\right]\right. \tag{90}$$

$$\left.+\bar{\xi}^2\left[\frac{3}{4}Q_2-2Q_2\left(\frac{Z_o}{g}\right)\omega_1^2\bar{r}^2+\left(\frac{1}{4}Q_5-\frac{3}{4}Q_6\right)\bar{r}^2\right]\right\}=0$$

Thus either

$$\bar{\xi} = 0$$

or

$$\bar{\xi}^2 = -\frac{\left[1-\bar{r}^2-2\left(\frac{Z_o}{g}\right)\omega_1^2\bar{r}^2-(8Q_7-4Q_8)P_1 Z_o\omega_1^2\bar{r}^4\right]}{\left[\frac{3}{4}Q_2-2Q_2\left(\frac{Z_o}{g}\right)\omega_1^2\bar{r}^2+\left(\frac{1}{4}Q_5-\frac{3}{4}Q_6\right)\bar{r}^2\right]} \tag{91}$$

The terms of P_1 which appear in equations (88) and (91) are due to the elastic effect of the bottom. For a rigid bottom these terms vanish.

<u>Harmonic Response</u>. For the determination of the harmonic response an approximate solution of equation (81) can be presented as

$$A_1(t) = \bar{\gamma} + \bar{\xi}\cos\Omega t \tag{92}$$

where $\bar{\xi}$ is of the first order and $\bar{\gamma}$ is of the second order.

Substituting equation (92) into equation (81) and neglecting terms of higher than third order yields

$$\bar{K}_3\bar{\xi}^3 + \bar{K}_2\bar{\xi}^2 + \bar{K}_1\bar{\xi} + \bar{K}_0 = 0 \tag{93}$$

where

$$\bar{K}_3 \equiv \Big\{ \left(\tfrac{3}{4}Q_2 - Q_1^2\right) + \left[\tfrac{1}{4}(Q_5 - 3Q_6) - Q_1\left(Q_3 - \tfrac{3}{2}Q_4\right)\right](\bar{r}^2) \tag{94}$$

$$+ \left[\tfrac{1}{2}Q_4(Q_3 - Q_4) + \left(\tfrac{3}{4}Q_1^2 - \tfrac{3}{2}Q_2\right)\left(\tfrac{Z_o}{g}\right)^2\omega_1^4\right](\bar{r}^4)$$

$$- \left[\left(\tfrac{3}{2}Q_2Q_7 - \tfrac{9}{8}Q_2Q_8\right)P_1(Z_o)\left(\tfrac{Z_o}{g}\right)\omega_1^4\right](\bar{r}^6) \Big\}$$

$$\bar{K}_2 \equiv \Big\{ \left[-\tfrac{1}{4}Q_1\left(\tfrac{Z_o}{g}\right)\omega_1^2\right](\bar{r}^2) + \left[\left(Q_1 - Q_4 + \tfrac{1}{2}Q_3\right)\left(\tfrac{Z_o}{g}\right)\omega^2 + Q_1\left(\tfrac{3}{2}Q_7\right.\right. \tag{95}$$

$$\left.\left. - Q_8\right)P_1(Z_o)\omega_1^2\right](\bar{r}^4) + \left[\tfrac{1}{2}(Q_7Q_3 - 2Q_7Q_4 + Q_4Q_8)P_1(Z_o)\omega_1^2\right](\bar{r}^6) \Big\}$$

$$\bar{K}_1 \equiv \Big\{ 1 - \bar{r}^2 - \tfrac{1}{2}\left(\tfrac{Z_o}{g}\right)^2\omega_1^4(\bar{r}^4) - \left[Q_7 - \tfrac{1}{2}Q_8 + Q_1(aX)\right]\left(\tfrac{Z_o}{g}\right)\omega_1^4(\bar{r}^6) \Big\} \tag{96}$$

$$\bar{K}_0 \equiv P_1(aX)(Z_o)\,\omega_1^2(\bar{r}^4) \tag{97}$$

and

$$\bar{Y} = \frac{1}{\left[1 - Q_1\left(\frac{Z_o}{g}\right)\omega_1^2\bar{r}^2\bar{\xi}\right]} \Big\{ \left[\tfrac{1}{2}\left(\tfrac{Z_o}{g}\right)\omega_1^2\bar{r}^2 + \tfrac{1}{2}(Q_7 - Q_8)P_1(Z_o\omega_1^2)(\bar{r}^4)\right]\bar{\xi} \tag{98}$$

$$- \left[\tfrac{1}{2}Q_1 + \tfrac{1}{2}(Q_3 - Q_4)(\bar{r}^2)\right]\bar{\xi}^2 + \left[\tfrac{3}{8}Q_2\left(\tfrac{Z_o}{g}\right)\omega_1^2\bar{r}^2\right]\bar{\xi}^3 \Big\}$$

with

$$\bar{r} = \left(\frac{\Omega}{\omega_1}\right) \tag{99}$$

If we let the plate rigidity approach infinity, then $P_1 \to 0$, and consequently, there will be no constant term in equation (93), i.e., $\bar{K}_0 = 0$, and equation (93) yields the response equation for the case of rigid bottom.

$$\bar{\xi}\,[\bar{K}_3\bar{\xi}^2 + \bar{K}_2\bar{\xi} + \bar{K}_1] = 0 \tag{100}$$

From this we find that either

$$\bar{\xi} = 0$$

or

$$\bar{K}_3\bar{\xi}^2 + \bar{K}_2\bar{\xi}^2 + \bar{K}_1 = 0 \tag{101}$$

Comparison of the results of the elastic bottom (equation (93)) with that of the rigid bottom case (equation (101)) shows that, in the case of an elastic bottom, a solution yielding a plane free surface does not exist in contrast to the liquid response in a container with rigid bottom. For the determination of $\bar{\gamma}$ from equation (98), the value $\bar{\xi}$ has to be obtained from equation (93). Thus the response curve may be presented in the $(\Omega\text{-}\eta^*)$ plane.

Coupled Motion (First and Second Axisymmetric Modes)

In actual liquid motion all modes come into play and are coupled with each other. This also could be observed in the experiments of an oscillating liquid. To obtain a theoretically better response of the liquid motion, two coupled modes will be considered in the analysis. The liquid motion will be considered as a primary mode plus a secondary mode. Since we are interested in the lower-frequency region, the first axisymmetric mode will be considered as a primary mode and the second axisymmetric mode will be a secondary mode.

The dynamic condition of the liquid free surface from equation (60) with n = 1, 2 is then given by

$$\{\dot{\alpha}_1(t) + (g+\ddot{Z}(t))\ a_1(t)\} + f_1 = 0 \tag{102}$$

where

$$f_1 \equiv \dot{\alpha}_o(t)\,a_1(t) + \lambda_1(\tanh\lambda_1 H)\alpha_o(t)\beta_1(t) + (0.3522803)\Big[\lambda_1(\tanh\lambda_1 H) a_1(t)\dot{\beta}_1(t) + \lambda_1^2\alpha_o(t)\alpha_1(t)\,a_1(t) + \frac{1}{2}\lambda_1^2(\tanh\lambda_1 H)^2\beta_1^2(t)\Big] + (0.1761056)\frac{1}{2}\cdot\lambda_1^2\alpha_1^2(t) + (0.2661406)\Big[\lambda_1(\tanh\lambda_1 H)\,a_2(t)\dot{\beta}_1(t) + \lambda_2(\tanh\lambda_2 H) a_1(t)\dot{\beta}_2(t) + \lambda_1\lambda_2(\tanh\lambda_1 H)(\tanh\lambda_2 H)\beta_1(t)\beta_2(t)\Big] + (0.2436875)\lambda_1\lambda_2 \alpha_1(t)\alpha_2(t) + (0.1379748)\lambda_1^3(\tanh\lambda_1 H)\alpha_1(t)\beta_1(t)\,a_1(t) + (0.4138998) \cdot\Big[\frac{1}{2}\lambda_1^2 a_1^2(t)\dot{\alpha}_1(t) + \lambda_1^3(\tanh\lambda_1 H)\alpha_1(t)\beta_1(t)\,a_1(t)\Big] \tag{103}$$

and

$$\{\dot{\alpha}_2(t) + (g+\ddot{Z}(t))\ a_2(t)\} + f_2 = 0 \tag{104}$$

where

$$f_2 \equiv \dot{\alpha}_o(t)\,a_2(t) + \lambda_2(\tanh\lambda_2 H)\alpha_o(t)\beta_2(t) + (0.4793192) \cdot\Big[\lambda_1(\tanh\lambda_1 H)\,a_1(t)\dot{\beta}_1(t) + \lambda_1^2\alpha_o(t)\alpha_1(t)\,a_1(t) + \frac{1}{2}\lambda_1^2(\tanh\lambda_1 H)^2\beta_1^2(t)\Big] \tag{105}$$

$$-(0.3241766)\frac{1}{2}\lambda_1^2\alpha_1^2(t)+(0.3019649)\Big[\lambda_1(\tanh\lambda_1 H)a_2(t)\dot{\beta}_1(t)$$

$$+\lambda_2(\tanh\lambda_2 H)a_1(t)\dot{\beta}_2(t)+\lambda_1\lambda_2(\tanh\lambda_1 H)(\tanh\lambda_2 H)\beta_1(t)\beta_2(t)\Big]$$

$$+(0.0824586)\lambda_1\lambda_2\alpha_1(t)\alpha_2(t)-(0.0185301)\lambda_1^3(\tanh\lambda_1 H)$$

$$\cdot\alpha_1(t)\beta_1(t)a_1(t)+(0.3156144)\Big[\frac{1}{2}\lambda_1^2 a_1^2(t)\dot{\alpha}_1(t)+\lambda_1^3$$

$$\cdot(\tanh\lambda_1 H)\alpha_1(t)\beta_1(t)a_1(t)\Big]$$

The kinematic condition of the liquid free surface from equation (62) is with $n = 1, 2$ given by

$$\{\dot{a}_1(t)-\lambda_1(\tanh\lambda_1 H)\,\beta_1(t)\}+g_1=0 \tag{106}$$

where

$$g_1 \equiv -(0.3522803)\lambda_1^2\alpha_1(t)a_1(t)+(0.1761056)\lambda_1^2 a_1(t)\alpha_1(t) \tag{107}$$

$$-(0.2661404)\Big[\lambda_1^2\alpha_1(t)a_2(t)+\lambda_2^2\alpha_2(t)a_1(t)\Big]+(0.2436875)\lambda_1\lambda_2$$

$$\Big[a_1(t)\alpha_2(t)+a_2(t)\alpha_1(t)\Big]-(0.0689751)\lambda_1^3(\tanh\lambda_1 H)\beta_1(t)a_1^2(t)$$

and

$$\{\dot{a}_2(t)-\lambda_2(\tanh\lambda_2 H)\,\beta_2(t)\}+g_2=0 \tag{108}$$

where

$$g_2 = -(0.4793192)\lambda_1^2 \alpha_1(t) a_1(t) - (0.3241766)\lambda_1^2 a_1(t)\alpha_1(t) \tag{109}$$

$$-(0.3019649)\left[\lambda_1^2 \alpha_1(t) a_2(t) + \lambda_2^2 \alpha_2(t) a_1(t)\right]$$

$$+(0.0824586)\left[\lambda_1\lambda_2 a_1(t)\alpha_2(t) + \lambda_1\lambda_2 a_2(t)\alpha_1(t)\right]$$

$$-(0.1763373)\lambda_1^3 (\tanh\lambda_1 H)\beta_1(t) a_1^2(t)$$

The relation between $\beta_n(t)$ and $\alpha_n(t)$ can be obtained from the kinematic condition of the container bottom, equations (74) and (75) as

$$\beta_1(t) - \alpha_1(t) = \delta_1(t) = \frac{a^5}{D(\sinh\lambda_1 H)}\left[(0.001906)\dot{q}_0(t) + (0.001211)\dot{q}_1(t)\right] \tag{110}$$

$$\beta_2(t) - \alpha_2(t) = \delta_2(t) = \frac{a^5}{D(\sinh\lambda_2 H)}\left[(0.000195)\dot{q}_0(t) + (0.000059)\dot{q}_2(t)\right] \tag{111}$$

$q_0(t)$, $q_1(t)$, and $q_2(t)$ are defined in equations (68), (69), and (70).

Combining equations (102), (104), (106), (108), (110), and (111) and neglecting higher order terms yields two coupled nonlinear ordinary differential equations in $a_1(t)$ and $a_2(t)$ with $A_1(t) = \frac{a_1(t)}{a}$ and $A_2(t) = \frac{a_2(t)}{a}$ and the excitation function $Z(t) = Z_0 \cos\Omega t$ we obtain the expressions:

$$\ddot{A}_1(t) + \omega_1^2\left(1 - \frac{Z_0}{g}\Omega^2\cos\Omega t\right)\left[A_1(t) + G_1 A_1^2(t) + G_2 A_1(t) A_2(t) \right. \tag{112}$$

$$\left. + G_3 A_1^3(t)\right] + G_4 \dot{A}_1^2(t) + G_5 A_1(t)\ddot{A}_1(t) + G_6 A_2(t)\ddot{A}_1(t) + G_7 A_1^2(t)\ddot{A}_1(t)$$

$$+ G_8 A_1(t)\ddot{A}_2(t) + G_9 \dot{A}_1(t)\dot{A}_2(t) + G_{10} A_1(t)\dot{A}_1^2(t) + G_{11} P_1 Z_o \Omega^4 \cos\Omega t\, A_1(t)$$

$$+ G_{12} P_1 Z_o \Omega^3 \sin\Omega t\, \dot{A}_1(t) = -(aXP_1 Z_o \Omega^4)\cos\Omega t$$

$$\ddot{A}_2(t) + \omega_2^2\left(1 - \frac{Z_o}{g}\Omega^2\cos\Omega t\right)\left[A_2(t) + H_1 A_1 A_2(t)\right] + H_2 \dot{A}_1^2(t) \tag{113}$$

$$+ H_3 A_1(t)\ddot{A}_1(t) + H_4 A_2 \ddot{A}_1(t) + H_5 A_1^2(t)\ddot{A}_1(t) + H_6 A_1(t)\ddot{A}_2(t) + H_7 \dot{A}_1(t)\dot{A}_2(t)$$

$$+ H_8 A_1(t)\dot{A}_1^2(t) + H_9 P_1 Z_o \Omega^4 \cos\Omega t\, A_1(t) + H_{10} P_1 Z_o \Omega^3 \sin\Omega t\, \dot{A}_1(t)$$

$$= -(aYP_2 Z_o \Omega^4)\cos\Omega t$$

The coefficients of the nonlinear terms G_1 through G_{12} and H_1 through H_{10} are listed in Appendix B. All of these coefficients are dependent upon the liquid height H and the tank radius a. The values X and Y are defined as $X = \lambda_1(\tanh\lambda_1 H)$ and $Y = \lambda_2(\tanh\lambda_2 H)$. $\omega_1^2 = \lambda_1 g(\tanh\lambda_1 H)$ and $\omega_2^2 = \lambda_2 g(\tanh\lambda_2 H)$ are the square of the first and second linearized axisymmetric natural frequencies of the liquid in a rigid container. The values of $\lambda_1 a$ and $\lambda_2 a$ are defined by equation (38). Furthermore,

$$P_1 = \frac{(0.001906)a^3(\rho_o H + \bar{\rho}\bar{h})}{D(\sinh\lambda_1 H)} \tag{114}$$

and

$$P_2 = \frac{(0.000195)a^3(\rho_o H + \bar{\rho}\bar{h})}{D(\sinh\lambda_2 H)}$$

P_1 and P_2 depend upon liquid density, liquid height, plate density, plate thickness, and its elastic property.

With increasing plate rigidity, D, the values of P_1 and P_2 approach zero. Then equations (112) and (113) coincide with the equations of motion for the case of a container with a rigid bottom. In the case of a tank with an elastic bottom, the nonhomogeneous terms in equations (112) and (113) become more significant for the larger tank diameters.

If equations (112) and (113) are linearized, they yield

$$\ddot{A}_1(t) + \omega_1^2\left(1 - \frac{Z_o}{g}\Omega^2\cos\Omega t\right)A_1(t) = -(aXP_1Z_o\Omega^4)\cos\Omega t \qquad (115)$$

$$\ddot{A}_2(t) + \omega_2^2\left(1 - \frac{Z_o}{g}\Omega^2\cos\Omega t\right)A_2(t) = -(aYP_2Z_o\Omega^4)\cos\Omega t \qquad (116)$$

These represent two uncoupled nonhomogeneous Mathieu equations. In the following analysis, the coupled one-half subharmonic and harmonic motions will be investigated.

<u>One-half Subharmonic Response</u>. According to the theory of the Mathieu equation and experimental observation, the first approximation of the coupled equations, (112) and (113), is written as

$$A_1(t) = \xi \sin \tfrac{1}{2}\Omega t + \zeta \sin \tfrac{3}{2}\Omega t \qquad (117)$$

$$A_2(t) = \gamma \cos\Omega t \qquad (118)$$

where ξ is assumed to be of first order, while ζ and γ are of second order.

Introducing these expressions into equations (112) and (113) and after some manipulation and truncation up to third order, the equation of the response function is given by

$$N_3 \xi^3 + N_1 \xi = 0 \tag{119}$$

where

$$N_1 \equiv \Big\{ 1 - \bar{r}^2 + 2\left(\frac{Z_o}{g}\right)\omega_1^2 \bar{r}^2 + (4G_{12} - 8G_{11}) P_1 Z_o \omega_1^2 \bar{r}^4 \tag{120}$$

$$+ \frac{1}{\left[1 - 4\left(\frac{\omega_1^2}{\omega_2^2}\right)\bar{r}^2\right]} \Big[8 P_2 G_2 (aY) Z_o \omega_1^2 \left(\frac{\omega_1^2}{\omega_2^2}\right) \bar{r}^4 + 16 P_2 (aY) Z_o \omega_1^2$$

$$\left(\frac{\omega_1^2}{\omega_2^2}\right)\left(G_9 - \frac{1}{2}G_6 - 2G_8 + 2G_2\left(\frac{Z_o}{g}\right)\omega_1^2\right)\bar{r}^6 \Big] \Big\}$$

$$N_3 \equiv \Big\{ \frac{3}{4}G_3 + \Big[2G_3\left(\frac{Z_o}{g}\right)\omega_1^2 + \frac{1}{4}(G_{10} - 3G_7) \Big] \bar{r}^2 \tag{121}$$

$$+ \frac{1}{\left[1 - 4\left(\frac{\omega_1^2}{\omega_2^2}\right)\bar{r}^2\right]} \Big[\frac{1}{4}G_2 (H_2 + H_3)\left(\frac{\omega_1^2}{\omega_2^2}\right)\bar{r}^2 - (H_2 + H_3)\Big(\frac{1}{4}G_6$$

$$+ G_8 - \frac{1}{2}G_9 - G_2\left(\frac{Z_o}{g}\right)\omega_1^2\Big)\left(\frac{\omega_1^2}{\omega_2^2}\right)\bar{r}^4 \Big]$$

$$+ \frac{1}{(1 - 9\bar{r}^2)\left[1 - 4\left(\frac{\omega_1^2}{\omega_2^2}\right)\bar{r}^2\right]} \left(m_1 + \frac{m_2}{\left[1 - 4\left(\frac{\omega_1^2}{\omega_2^2}\right)\bar{r}^2\right]} \right) \Big\}$$

and

$$m_1 \equiv \left[-2\left(\frac{Z_o}{g}\right)\omega_1^2 \bar{r}^2 + (8G_{11} - 12G_{12}) P_1 Z_o \omega_1^2 \bar{r}^4 \right] \Big\{ \frac{1}{4}G_3 \tag{122}$$

$$+ \Big[\left(\frac{1}{4}G_2 H_2 + \frac{1}{4}G_2 H_3 - G_3\right)\left(\frac{\omega_1^2}{\omega_2^2}\right) - \Big(\frac{1}{4}G_7 + \frac{1}{4}G_{10}$$

$$-\frac{3}{2}G_3\left(\frac{Z_o}{g}\right)\omega_1^2\Big]\bar{r}^2+\Big[G_7+G_{10}-6G_3\left(\frac{Z_o}{g}\right)\omega_1^2-(H_2+H_3)$$

$$\cdot\Big(\frac{1}{4}G_6+G_8+\frac{1}{2}G_9-\frac{1}{2}G_2\left(\frac{Z_o}{g}\right)\omega_1^2\Big]\left(\frac{\omega_1^2}{\omega_2^2}\right)\bar{r}^4\Big\}$$

$$m_2 \equiv \Big[\frac{1}{2}G_2(3H_2-5H_3)\left(\frac{\omega_1^2}{\omega_2^2}\right)\bar{r}^2-\Big(\frac{1}{2}G_6+2G_8-G_9-2G_2\left(\frac{Z_o}{g}\right)\omega_1^2\Big) \tag{123}$$

$$\cdot(3H_2-5H_3)\left(\frac{\omega_1^2}{\omega_2^2}\right)\bar{r}^4\Big]\Big\{2\left(\frac{Z_o}{g}\right)\omega_1^2\bar{r}^2-\Big[8\left(\frac{Z_o}{g}\right)\omega_1^2\left(\frac{\omega_1^2}{\omega_2^2}\right)+(8G_{11}$$

$$+4G_{12})P_1Z_o\omega_1^2-8G_2P_2(aY)Z_o\omega_1^2\left(\frac{\omega_1^2}{\omega_2^2}\right)\Big]\bar{r}^4+\Big[(16G_{11}$$

$$+8G_{12})P_1-\Big(4G_6+16G_8+8G_9-8G_2\left(\frac{Z_o}{g}\right)\omega_1^2\Big)$$

$$\cdot P_2(aY)\Big]Z_o\omega_1^2\left(\frac{\omega_1^2}{\omega_2^2}\right)\bar{r}^6\Big\}$$

Furthermore,

$$\bar{r} = \left(\frac{\frac{\Omega}{2}}{\omega_1}\right), \tag{124}$$

$$Y = \frac{-1}{\left[1-4\left(\frac{\omega_1^2}{\omega_2^2}\right)\bar{r}^2\right]}\Big\{16P_2(aY)Z_o\omega_1^2\left(\frac{\omega_1^2}{\omega_2^2}\right)\bar{r}^4+\xi^2\left(\frac{1}{2}H_2+\frac{1}{2}H_3\right)\left(\frac{\omega_1^2}{\omega_2^2}\right)\bar{r}^2 \tag{125}$$

$$+\frac{1}{(1-9\bar{r}^2)\left[1-4\left(\frac{\omega_1^2}{\omega_2^2}\right)\bar{r}^2\right]}\Big(2\left(\frac{Z_o}{g}\right)\omega_1^2\bar{r}^4-\Big[8\left(\frac{Z_o}{g}\right)\omega_1^2\left(\frac{\omega_1^2}{\omega_2^2}\right)+(8G_{11}+4G_{12})$$

$$\cdot P_1Z_o\omega_1^2-8G_2P_2(aY)Z_o\omega_1^2\left(\frac{\omega_1^2}{\omega_2^2}\right)\Big]\bar{r}^6+\Big[(16G_{11}+8G_{12})P_1-(4G_6$$

$$+16G_8+8G_9-8G_2\left(\frac{Z_o}{g}\right)\omega_1^2)P_2(aY)\Big]Z_o\omega_1^2\left(\frac{\omega_1^2}{\omega_2^2}\right)\bar{r}^8\Big)\xi^2\Big\}$$

and

$$\zeta = \frac{-1}{(1-9\bar{r}^2)}\Big\{\Big[-2\Big(\frac{Z_o}{g}\Big)\omega_1^2\bar{r}^2+(8G_{11}+4G_{12})P_1 Z_o\omega_1^2\bar{r}^4\Big]\xi \tag{126}$$

$$+\Big[\frac{1}{2}G_2+\Big(G_2\Big(\frac{Z_o}{g}\Big)\omega_1^2-\frac{1}{2}G_6-2G_8-G_9\Big)(\bar{r}^2)\Big]\xi\gamma+\Big[-\frac{1}{4}G_3$$

$$+\Big(\frac{1}{4}G_7+\frac{1}{4}G_{10}-\frac{3}{2}G_3\Big(\frac{Z_o}{g}\Big)\omega_1^2\Big)\bar{r}^2\Big]\xi^3\Big\}$$

The solution of equation (119) is either

$$\xi = 0$$

or

$$\xi^2 = -\frac{N_1}{N_3} \tag{127}$$

After ξ is obtained, then γ and ζ may be determined from equations (125) and (126).

The second approximate solution for the above coupled nonlinear ordinary differential equations, (112) and (113), may be written as

$$A_1(t) = \bar{\xi}\cos\tfrac{1}{2}\Omega t + \bar{\zeta}\cos\tfrac{3}{2}\Omega t \tag{128}$$

$$A_2(t) = \bar{\gamma}\cos\Omega t \tag{129}$$

giving finally the equation of the response function

$$\bar{N}_3\bar{\xi}^3 + \bar{N}_1\bar{\xi} = 0 \tag{130}$$

where

$$\bar{N}_1 \equiv \Big\{1-\bar{r}^2-2\left(\frac{Z_o}{g}\right)\omega_1^2\bar{r}^2-(4G_{12}-8G_{11})P_1 Z_o\omega_1^2\bar{r}^4 \tag{131}$$

$$-\frac{1}{\left[1-4\left(\frac{\omega_1^2}{\omega_2^2}\right)\bar{r}^2\right]}\Big[8P_2G_2(aY)Z_o\omega_1^2\left(\frac{\omega_1^2}{\omega_2^2}\right)\bar{r}^4+16P_2(aY)$$

$$\cdot Z_o\omega_1^2\left(\frac{\omega_1^2}{\omega_2^2}\right)\left(G_9-\frac{1}{2}G_6-2G_8-2G_2\left(\frac{Z_o}{g}\right)\omega_1^2\right)\bar{r}^6\Big]\Big\}$$

$$\bar{N}_3 \equiv \Big\{\frac{3}{4}G_3+\left[-2G_3\left(\frac{Z_o}{g}\right)\omega_1^2+\frac{1}{4}(G_{10}-3G_7)\right]\bar{r}^2+\frac{1}{\left[1-4\left(\frac{\omega_1^2}{\omega_2^2}\right)\bar{r}^2\right]} \tag{132}$$

$$\cdot(H_2+H_3)\Big[\frac{1}{4}G_2\left(\frac{\omega_1^2}{\omega_2^2}\right)\bar{r}^2-\left(\frac{1}{4}G_6+G_8-\frac{1}{2}G_9+G_2\left(\frac{Z_o}{g}\right)\omega_1^2\right)$$

$$\left(\frac{\omega_1^2}{\omega_2^2}\right)\bar{r}^4\Big]+\frac{1}{(1-9\bar{r}^2)\left[1-4\left(\frac{\omega_1^2}{\omega_2^2}\right)\bar{r}^2\right]}\left(\bar{m}_1+\frac{\bar{m}_2}{\left[1-4\left(\frac{\omega_1^2}{\omega_2^2}\right)\bar{r}^2\right]}\right)\Big\}$$

and

$$\bar{m}_1 \equiv \left[2\left(\frac{Z_o}{g}\right)\omega_1^2\bar{r}^2-(8G_{11}-12G_{12})P_1Z_o\omega_1^2\bar{r}^4\right]\Big\{\frac{1}{4}G_3+\Big[\Big(\frac{1}{4}G_2H_2 \tag{133}$$

$$+\frac{1}{4}G_2H_3-G_3\Big)\left(\frac{\omega_1^2}{\omega_2^2}\right)-\left(\frac{1}{4}G_7+\frac{1}{4}G_{10}+\frac{3}{2}G_3\left(\frac{Z_o}{g}\right)\omega_1^2\right)\Big]\bar{r}^2$$

$$+\Big[(G_7+G_{10})+6G_3\left(\frac{Z_o}{g}\right)\omega_1^2-(H_2+H_3)\Big(\frac{1}{4}G_6+G_8+\frac{1}{2}G_9$$

$$+\frac{1}{2}G_2\left(\frac{Z_o}{g}\right)\omega_1^2\Big)\Big]\left(\frac{\omega_1^2}{\omega_2^2}\right)\bar{r}^4\Big\}$$

$$\bar{m}_2 \equiv \Big[\frac{1}{2}G_2(3H_2-5H_3)\left(\frac{\omega_1^2}{\omega_2^2}\right)\bar{r}^2-\left(\frac{1}{2}G_6+2G_8-G_9+2G_2\left(\frac{Z_o}{g}\right)\omega_1^2\right) \tag{134}$$

$$\cdot(3H_2-5H_3)\left(\frac{\omega_1^2}{\omega_2^2}\right)\bar{r}^4\Big]\Big\{-2\left(\frac{Z_o}{g}\right)\omega_1^2\bar{r}^2+\Big[8\left(\frac{Z_o}{g}\right)\omega_1^2\left(\frac{\omega_1^2}{\omega_2^2}\right)$$

$$+(8G_{11}+4G_{12})P_1 Z_0\omega_1^2 - 8G_2P_2(aY)Z_0\omega_1^2(\frac{\omega_1^2}{\omega_2^2})]\bar{r}^4$$

$$-[(16G_{11}+8G_{12})P_1-(4G_6+16G_8+8G_9+8G_2(\frac{Z_0}{g})\omega_1^2)$$

$$P_2(aY)]Z_0\omega_1^2(\frac{\omega_1^2}{\omega_2^2})\bar{r}^6\}$$

Furthermore,

$$\bar{\gamma} = \frac{-1}{[1-4(\frac{\omega_1^2}{\omega_2^2})\bar{r}^2]}\Big\{-16P_2(aY)Z_0\omega_1^2(\frac{\omega_1^2}{\omega_2^2})\bar{r}^4+\bar{\xi}^2(\frac{1}{2}H_2+\frac{1}{2}H_3) \qquad (135)$$

$$(\frac{\omega_1^2}{\omega_2^2})\bar{r}^2+\frac{1}{(1-9\bar{r}^2)[1-4(\frac{\omega_1^2}{\omega_2^2})\bar{r}^2]}\Big(-2(\frac{Z_0}{g})\omega_1^2\bar{r}^4+[8(\frac{Z_0}{g})\omega_1^2(\frac{\omega_1^2}{\omega_2^2})+(8G_{11}$$

$$+4G_{12})P_1Z_0\omega_1^2-8G_2P_2(aY)Z_0\omega_1^2(\frac{\omega_1^2}{\omega_2^2})]\bar{r}^6-[(16G_{11}+8G_{12})P_1-(4G_6$$

$$+16G_8+8G_9+8G_2(\frac{Z_0}{g})\omega_1^2)P_2(aY)]Z_0\omega_1^2(\frac{\omega_1^2}{\omega_2^2})\bar{r}^8\Big)\bar{\xi}^2\Big\}$$

and

$$\bar{\zeta} = \frac{-1}{(1-9\bar{r}^2)}\Big\{[2(\frac{Z_0}{g})\omega_1^2\bar{r}^2-(8G_{11}+4G_{12})P_1Z_0\omega_1^2\bar{r}^4]\bar{\xi}+[\frac{1}{2}G_2 \qquad (136)$$

$$+(-G_2(\frac{Z_0}{g})\omega_1^2-\frac{1}{2}G_6-2G_8-G_9)\bar{r}^2]\bar{\xi}\bar{\gamma}+[-\frac{1}{4}G_3+(\frac{1}{4}G_7$$

$$+\frac{1}{4}G_{10}+\frac{3}{2}G_3(\frac{Z_0}{g})\omega_1^2)\bar{r}^2]\bar{\xi}^3\Big\}$$

The response function is either

$$\bar{\xi} = 0$$

or

$$\bar{\xi}^2 = -\frac{\bar{N}_1}{\bar{N}_3} \tag{137}$$

The values of $\bar{\gamma}$ and $\bar{\zeta}$ may be obtained from equations (135) and (136) by introducing the value for $\bar{\xi}$.

Harmonic Response. For the harmonic response, an approximate solution of equations (112) and (113) may be written as

$$A_1(t) = \bar{\xi}\cos\Omega t + \bar{\gamma} \tag{138}$$

$$A_2(t) = \bar{\zeta}\cos 2\Omega t \tag{139}$$

where we assume $\bar{\xi}$ is to be of first order, and $\bar{\gamma}$ and $\bar{\zeta}$ are of second order.

By substituting these solutions into equations (112) and (113) and neglecting terms of higher than third order, the equation for the response function is obtained. Thus,

$$\bar{M}_3\bar{\xi}^3 + \bar{M}_2\bar{\xi}^2 + \bar{M}_1\bar{\xi} + \bar{M}_0 = 0 \tag{140}$$

where

$$\bar{M}_3 \equiv \Big\{\left(\frac{3}{4}G_3 - G_1^2\right) + \Big[\frac{1}{4}G_{10} - \frac{3}{4}G_7 - G_1\left(G_4 - \frac{3}{2}G_5\right) + \frac{1}{4}G_2(H_2 \tag{141}$$

$$+H_3) - (3G_3 - 4G_1^2)\left(\frac{\omega_1^2}{\omega_2^2}\right)\Big]\bar{r}^2 + \Big[-\frac{1}{2}G_5(G_5 - G_4) + (-G_{10} + 3G_7$$

$$-6G_1G_5 + 4G_1G_4 - \frac{1}{2}(H_2 + H_3)\left(\frac{1}{2}G_6 + 2G_8 - G_9\right))\left(\frac{\omega_1^2}{\omega_2^2}\right) + \left(-\frac{3}{8}\right.$$

$$-\frac{1}{4}G_1H_1-\frac{9}{8}G_3+\frac{3}{4}G_1\left(\frac{1}{2}H_1+G_1\right)\Big)\left(\frac{Z_0}{g}\right)^2\omega_1^4\Big]\bar{r}^4+\Big[-2G_5(G_4-G_5)$$

$$\left(\frac{\omega_1^2}{\omega_2^2}\right)+\Big(-\frac{1}{2}G_1H_1-\frac{1}{4}H_1(G_4-G_5)+\frac{1}{4}G_5H_1+\Big(-\frac{1}{8}G_2H_2-\frac{1}{8}G_2H_3$$

$$+\frac{3}{2}+\frac{9}{2}G_3-3G_1^2\Big)\left(\frac{\omega_1^2}{\omega_2^2}\right)\left(\frac{Z_0}{g}\right)^2\omega_1^4+\Big(G_{11}\left(\frac{3}{8}+\frac{1}{4}G_1H_1\right)-\frac{1}{2}G_1H_1$$

$$(G_{12}-G_{11})-\frac{9}{8}G_3(G_{12}-G_{11})\Big)P_1Z_0\left(\frac{Z_0}{g}\right)\omega_1^4\Big]\bar{r}^6+\Big[\frac{1}{4}G_{11}H_1(G_4$$

$$-G_5)-\frac{1}{4}G_5H_1(G_{11}-G_{12})+\Big(\frac{1}{8}G_{11}G_2(H_2+H_3)+\frac{1}{8}G_2G_5$$

$$\cdot(H_9+H_{10})-\frac{3}{2}G_{11}+\frac{9}{2}G_3(G_{12}-G_{11})-\frac{1}{2}G_1(H_9+H_{10})$$

$$\cdot\left(\frac{1}{2}G_6+2G_8-G_9\right)\Big)\left(\frac{\omega_1^2}{\omega_2^2}\right)\Big]Z_0\left(\frac{Z_0}{g}\right)\omega_1^4\bar{r}^8\Big\}$$

$$\bar{M}_2 = \Big\{\Big[-\left(\frac{1}{4}G_1+\frac{1}{2}H_1\right)\left(\frac{Z_0}{g}\right)\omega_1^2\Big]\bar{r}^2+\Big[\Big(G_1+\frac{1}{2}H_1+\frac{1}{2}G_4-G_5 \tag{142}$$

$$+G_1\left(\frac{\omega_1^2}{\omega_2^2}\right)\Big)\left(\frac{Z_0}{g}\right)\omega_1^2+\Big(G_1\left(G_{12}-\frac{3}{2}G_{11}\right)-\frac{1}{4}G_2(H_9+H_{10})\left(\frac{\omega_1^2}{\omega_2^2}\right)\Big)P_1Z_0$$

$$\omega_1^2\Big]\bar{r}^4+\Big[\Big(-4G_1+2P_1G_{11}(G_5-G_4)+2G_5\Big)\left(\frac{Z_0}{g}\right)\omega_1^2\left(\frac{\omega_1^2}{\omega_2^2}\right)+\Big(G_{11}G_5$$

$$-\frac{1}{2}G_{11}G_4-\frac{1}{2}G_5G_{12}\Big)P_1Z_0\omega_1^2+\Big(6G_1G_{11}-4G_1G_{12}+\frac{1}{2}(H_9$$

$$+H_{10})\left(\frac{1}{2}G_6+2G_8-G_9\right)\Big)\left(\frac{\omega_1^2}{\omega_2^2}\right)P_1Z_0\omega_1^2\Big]\bar{r}^6+\Big[\Big(2G_4G_{11}$$

$$-4G_5G_{11}+2G_5G_{12}\Big)P_1Z_0\omega_1^2\left(\frac{\omega_1^2}{\omega_2^2}\right)\Big]\bar{r}^8\Big\}$$

$$\bar{M}_1 \equiv \Big\{ 1 - [1 + 4(\frac{\omega_1^2}{\omega_2^2})]\bar{r}^2 + [4(\frac{\omega_1^2}{\omega_2^2}) - \frac{1}{2}(\frac{Z_o}{g})^2\omega_1^4]\bar{r}^4 \tag{143}$$

$$+ [-\frac{1}{2} G_{12} P_1 Z_o (\frac{Z_o}{g})\omega_1^4 + 2(\frac{\omega_1^2}{\omega_2^2})(\frac{Z_o}{g})^2\omega_1^4]\bar{r}^6$$

$$+ [2(G_{12} - G_{11}) P_1 Z_o (\frac{Z_o}{g})\omega_1^4(\frac{\omega_1^2}{\omega_2^2})]\bar{r}^8 \Big\}$$

$$\bar{M}_0 \equiv \Big\{ (aX) P_1 Z_o \omega_1^2 \bar{r}^4 - 4(aX) P_1 Z_o \omega_1^2 (\frac{\omega_1^2}{\omega_2^2})\bar{r}^6 \Big\} \tag{144}$$

Furthermore,

$$\bar{r} = \left(\frac{\Omega}{\omega_1}\right)$$

$$\bar{\zeta} = \frac{1}{\Big\{[1 - 4(\frac{\omega_1^2}{\omega_2^2})\bar{r}^2] - \frac{1}{2}H_1(\frac{Z_o}{g})\omega_1^2\bar{r}^2\bar{\xi}\Big\}} \Big\{ -\bar{\xi}[\frac{1}{2}(H_9 + H_{10}) P_1 Z_o \omega_1^2 \tag{145}$$

$$(\frac{\omega_1^2}{\omega_2^2})\bar{r}^4] + \bar{\xi}^2[\frac{1}{2}(H_2 + H_3)(\frac{\omega_1^2}{\omega_2^2})\bar{r}^2] \Big\}$$

and

$$\bar{Y} = \frac{1}{\Big\{1 + \bar{\xi}[-G_1(\frac{Z_o}{g})\omega_1^2\bar{r}^2] + \bar{\xi}^2[\frac{3}{2}G_3 + (\frac{1}{2}G_{10} - G_7)\bar{r}^2]\Big\}} \Big\{ \frac{1}{4}G_2 \tag{146}$$

$$\cdot(\frac{Z_o}{g})\omega_1^2\bar{r}^2\bar{\xi}\bar{\zeta} + [\frac{1}{2}(\frac{Z_o}{g})\omega_1^2\bar{r}^2 + \frac{1}{2}(G_{12} - G_{11}) P_1 Z_o \omega_1^2 \bar{r}^4]\bar{\xi} - [\frac{1}{2}G_1$$

$$+ \frac{1}{2}(G_4 - G_5)\bar{r}^2]\bar{\xi}^2 + \frac{3}{8}(\frac{Z_o}{g})\omega_1^2\bar{r}^2\bar{\xi}^3 \Big\}$$

If the tank bottom is rigid, then $M_o = 0$ and then equation (140) yields

$$\bar{\xi}\,[\bar{M}_{30}\bar{\xi}^2 + \bar{M}_{20}\bar{\xi} + \bar{M}_{10}] = 0 \tag{147}$$

and exhibits the form of the equation for the response of the liquid system with a rigid bottom. The values $\overline{M}_{1o}$, $\overline{M}_{2o}$, and $\overline{M}_{3o}$ are obtained from the above expressions (141) through (143) by setting $P_1 = 0$. The solution of this equation is either

$$\overline{\xi} = 0$$

or

$$\overline{M}_{3o}\overline{\xi}^2 + \overline{M}_{2o}\overline{\xi} + \overline{M}_{1o} = 0 \tag{148}$$

The difference of the response of the liquid in a container with a rigid bottom or an elastic bottom may be summarized in the following way. In the case of a rigid bottom, the response is either zero (plane free surface) or it exhibits a finite amplitude. In the case of a container with an elastic bottom, however, a finite amplitude response is always present. After the value $\overline{\xi}$ has been obtained from equation (140), the expressions for $\overline{\zeta}$ and $\overline{\gamma}$ may be determined from equations (145) and (146).

CHAPTER V

FREE SURFACE ELEVATION, FORCES AND MOMENTS

Liquid Free Surface

For a given value of the forcing frequency Ω and the excitation amplitude Z_o, the free surface amplitude for an axisymmetric case can be approximately calculated from the following expressions.

One-half Subharmonic Response

Uncoupled Mode.

$$\eta = a_1(t)J_o(\lambda_1 r) = a\ A_1(t)J_o(\lambda_1 r) \tag{149}$$

$$= (a\ \xi \sin \tfrac{1}{2}\Omega t)J_o(\lambda_1 r)$$

where a is the tank radius and ξ is the response amplitude which was obtained from equation (88).

Coupled Modes.

$$\eta = a_1(t)J_o(\lambda_1 r) + a_2(t)J_o(\lambda_2 r) \tag{150}$$

$$= a(\xi \sin \tfrac{1}{2}\Omega t + \zeta \sin \tfrac{3}{2}\Omega t)J_o(\lambda_1 r)$$

$$+ a(\gamma \cos\Omega t)J_o(\lambda_2 r)$$

where a is the tank radius, ξ, γ, and ζ are the amplitudes which were obtained from equations (119), (125), and (126).

In the case of a one-half subharmonic response, the second set of solutions for both uncoupled and coupled motion correspond to unstable motions which were not observed in the experiments.

Harmonic Response

The approximation for the elevation of the liquid free surface for harmonic liquid motions for both uncoupled and coupled motion can be expressed as follows.

Uncoupled Mode.

$$\eta = a(\overline{\gamma} + \overline{\xi} \cos\Omega t)J_0(\lambda_1 r) \tag{151}$$

where a is the tank radius, $\overline{\xi}$ and $\overline{\gamma}$ are the amplitudes which were obtained from equations (93) and (98).

Coupled Mode.

$$\eta = a(\overline{\xi} \cos\Omega t + \overline{\gamma})J_0(\lambda_1 r) + a(\overline{\zeta} \cos 2\Omega t)J_0(\lambda_2 r) \tag{152}$$

where a is the tank radius, $\overline{\xi}$, $\overline{\zeta}$, and $\overline{\gamma}$ are the amplitudes which were obtained from equations (140), (145), and (146).

An approximate expression for the deflection of the elastic bottom $w(r,t)$ can also be obtained from equation (71) for coupled or uncoupled liquid motion as

$$w(r,t) = \frac{1}{D}\left\{ \frac{q_0(t)}{64}\left(a^2 - r^2\right)^2 + \frac{q_1(t)}{\lambda_1^4}\left(J_0(\lambda_1 r) - J_0(\lambda_1 a)\right) \right.$$

$$\left. + \frac{q_2(t)}{\lambda_2^4}\left(J_0(\lambda_2 r) - J_0(\lambda_2 a)\right)\right\}$$

where $q_0(t)$, $q_1(t)$, and $q_2(t)$ are defined in equations (68), (69), and (70). For their determination, one needs the terms $A_1(t)$ and $A_2(t)$ which are obtained from equation (81) for uncoupled and from equations (112) and (113) for the coupled liquid motion. It should be pointed out here that, in order to get approximate expressions for $\beta_1(t) - \alpha_1(t)$ and $\beta_2(t) - \alpha_2(t)$ from equation (72), the assumption has been made that the difference between $\beta_n(t)$ and $\alpha_n(t)$ for each n is a small value such that all higher order terms on the left hand side of equation (72) may be neglected. A more accurate expression could be obtained for $\beta_n(t) - \alpha_n(t)$ from equation (72) by employing an iteration procedure, i.e., using the results which have been obtained in the analysis as a first approximation. Substituting these results of $\alpha_n(t)$ and $\beta_n(t)$ (which are consequently in terms of $a_n(t)$) into the left hand side of equation (72) and retaining higher order terms, we obtain a more accurate expression for $\beta_n(t) - \alpha_n(t)$.

In general, the difference between $\beta_n(t)$ and $\alpha_n(t)$ is of a small magnitude only if the ratio of liquid height H to the radius a is greater than 1 ($\frac{H}{a} \geqq 1$), or if the container bottom is of moderate thickness, for example, $\frac{\bar{h}}{a} \geqq \frac{1}{200}$. For $\frac{H}{a} \geqq \frac{1}{2}$, the value of $\beta_1(t) - \alpha_1(t) \sim 10^{-3}$ or less for different bottom thicknesses. For the case of a rigid bottom, $\beta_n(t)$ is equal to $\alpha_n(t)$.

Forces and Moments

Since the forces and moments exerted by the liquid motion play an important role in the overall stability behavior of the space vehicle, therefore, the components in the x, y, z directions shall be presented.

The pressure distribution is given by Bernoulli's unsteady flow equation as

$$p = - \rho_0\left[\frac{\partial\Phi}{\partial t} + (g+\ddot{Z}(t))z + \tfrac{1}{2}(\nabla\Phi)^2\right] \tag{153}$$

The velocity potential Φ from equation (13) is substituted into equation (153) to obtain

$$\begin{aligned} p = - \rho_0\Big\{ & \dot{\alpha}_{00}(t)z + \dot{\beta}_{00}(t) + \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}(\sinh\lambda_{mn}H)\Big[\dot{\alpha}_{mn}(t)\frac{\cosh\lambda_{mn}z}{\sinh\lambda_{mn}H} \\ & + \dot{\beta}_{mn}(t)\frac{\sinh\lambda_{mn}z}{\cosh\lambda_{mn}H}\Big]J_m(\lambda_{mn}r)\cos m\theta + (g + \ddot{Z}(t))z \\ & + \frac{1}{2}\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\sum_{p=0}^{\infty}\sum_{q=1}^{\infty}(\sinh\lambda_{mn}H)(\sinh\lambda_{pq}H)\Big[\lambda_{mn}\lambda_{pq}\Big(\alpha_{mn}(t) \\ & \frac{\cosh\lambda_{mn}z}{\sinh\lambda_{mn}H} + \beta_{mn}(t)\frac{\sinh\lambda_{mn}z}{\cosh\lambda_{mn}H}\Big)\Big(\alpha_{pq}(t)\frac{\cosh\lambda_{pq}z}{\sinh\lambda_{pq}H} + \beta_{pq}(t) \\ & \frac{\sinh\lambda_{pq}z}{\cosh\lambda_{pq}H}\Big)J_m'(\lambda_{mn}r)J_p'(\lambda_{pq}r)\cos m\theta\cos p\theta + \frac{1}{r^2}mp \\ & \cdot\Big(\alpha_{mn}(t)\frac{\cosh\lambda_{mn}z}{\sinh\lambda_{mn}H} + \beta_{mn}(t)\frac{\sinh\lambda_{mn}z}{\cosh\lambda_{mn}H}\Big)\Big(\alpha_{pq}(t)\frac{\cosh\lambda_{pq}z}{\sinh\lambda_{pq}H} + \beta_{pq}(t) \\ & \cdot\frac{\sinh\lambda_{pq}z}{\cosh\lambda_{pq}H}\Big)J_m(\lambda_{mn}r)J_p(\lambda_{pq}r)\sin m\theta\sin p\theta + \lambda_{mn}\lambda_{pq}\Big(\alpha_{mn}(t) \end{aligned} \tag{154}$$

$$\frac{\sinh\lambda_{mn}z}{\sinh\lambda_{mn}H}+\beta_{mn}(t)\frac{\cosh\lambda_{mn}z}{\cosh\lambda_{mn}H}\Big)\Big(\alpha_{pq}(t)\frac{\sinh\lambda_{pq}z}{\sinh\lambda_{pq}H}+\beta_{pq}(t)\frac{\cosh\lambda_{pq}z}{\cosh\lambda_{pq}H}\Big)$$

$$J_m(\lambda_{mn}r)J_p(\lambda_{pq}r)\cos m\theta\cos p\theta\Big]+\frac{1}{2}\alpha_{oo}^2(t)$$

$$+\alpha_{oo}(t)\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\lambda_m(\sinh\lambda_{mn}H)\Big(\alpha_{mn}(t)\frac{\sinh\lambda_{mn}z}{\sinh\lambda_{mn}H}$$

$$+\beta_{mn}(t)\frac{\cosh\lambda_{mn}z}{\cosh\lambda_{mn}H}\Big)J_m(\lambda_{mn}r)\cos m\theta\Big\}$$

If $\hat{e}_n$ is a unit vector outward normal to the tank walls and $\hat{e}_q$ is a unit vector in the direction for which the liquid force is required, then

$$F_q = \int_S p\ \hat{e}_n\cdot\hat{e}_q dS \tag{155}$$

where the integration is performed over the wetted surface.

For the circular cylindrical container, the forces acting on the container walls due to liquid oscillations are

$$\begin{bmatrix}F_x\\F_y\end{bmatrix} = \int_o^{2\pi}\int_{-H}^{\eta|_{wall}} ap\Big|_{r=a}\begin{bmatrix}\cos\theta\\\sin\theta\end{bmatrix}d\theta dz \tag{156}$$

The force in the z direction is due to the pressure distribution on the tank bottom and is obtained from

$$F_z = -\int_o^{2\pi}\int_o^a p\Big|_{z=-H} r d\theta dr \tag{157}$$

The liquid moment about axes parallel to the x, y, z axes and through the point (0, 0, $-\frac{1}{2}$H) are

$$\begin{bmatrix} M_x \\ M_y \end{bmatrix} = \int_0^{2\pi}\int_{-H}^{\eta|_{wall}} \left(\frac{H}{2} + z\right)\left(ap\Big|_{r=a}\right)\begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix} d\theta dz \tag{158}$$

$$+ \int_0^{2\pi}\int_0^{a} r^2 p\Big|_{z=-H} \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix} d\theta dr$$

and

$$M_z = 0. \tag{159}$$

An approximation of the forces and moments is obtained by using the linearized Bernoulli equation. The fluid forces and moments are then

$$F_x = \rho_o(\pi a^2)\sum_{n=1}^{\infty}(\sinh\lambda_{1n}H)\frac{1}{(\lambda_{1n}a)}\Big\{\dot{\beta}_{1n}(t) - \dot{\alpha}_{1n}(t) \tag{160}$$

$$- \left[\dot{\alpha}_{1n}(t)\frac{\sinh\lambda_{1n}\eta|_{r=a}}{\sinh\lambda_{1n}H} + \dot{\beta}_{1n}(t)\frac{\cosh\lambda_{1n}\eta|_{r=a}}{\cosh\lambda_{1n}H}\right]\Big\} J_1(\lambda_{1n}a) \quad ,$$

$$F_y = 0, \tag{161}$$

$$F_z = -\left(\rho_o \pi a^2 H\right)\left(g + \ddot{Z}(t)\right) , \tag{162}$$

$$M_x = 0, \tag{163}$$

and

$$M_y = (\rho_o \pi a)\sum_{n=1}^{\infty} J_1(\lambda_{1n}a)\Big\{\dot{\alpha}_{1n}(t)\Big[\frac{1}{\lambda_{1n}^2}(1-\cosh\lambda_{1n}H)+\frac{H}{2}\Big(\frac{1}{\lambda_{1n}}\Big) \tag{164}$$

$$\sinh\lambda_{1n}H\Big] + \dot{\beta}_{1n}(t)(\tanh\lambda_{1n}H)\Big[\frac{1}{\lambda_{1n}^2}\sinh\lambda_{1n}H - \frac{H}{2}\Big(\frac{1}{\lambda_{1n}}\Big)(1+\cosh\lambda_{1n}H)$$

$$\Big]\Big\} + (\rho_o \pi a^2)(\sinh\lambda_{1n}H)\Big\{\dot{\beta}_{1n}(t)\tanh\lambda_{1n}H - \dot{\alpha}_{1n}(t)\coth\lambda_{1n}H\Big\}\Big(\frac{1}{\lambda_{1n}}\Big)J_2(\lambda_{2}a).$$

The forces and moments due to the liquid motion, which have been derived as shown in equations (160) through (164), can be expressed as free surface displacements $a_{mn}(t)$ by using the dynamical condition of the liquid free surface and the relations for $\beta_{mn}(t)$ and $\alpha_{mn}(t)$ which were derived in equation (36) by considering the motion of the elastic bottom associated with its kinematic condition. It is from equations (23) and (36).

$$\dot{\alpha}_{mn}(t) + (g+\ddot{Z}(t))\, a_{mn}(t) = 0 \tag{165}$$

where the nonlinear terms have been neglected.

$$\beta_{mn}(t) - \alpha_{mn}(t) = \delta_{mn}(t) \tag{166}$$

where $\delta_{mn}(t)$ are functions of the excitation parameters and the elastic property of the tank bottom. With $Z(t) = Z_0 \cos\Omega t$ and $A_{mn}(t) = \frac{a_{mn}(t)}{a}$, thus

$$\dot{\alpha}_{mn}(t) = -(g - Z_0\Omega^2\cos\Omega t)\, A_{mn}(t)a \tag{167}$$

and

$$\dot{\beta}_{mn}(t) = \dot{\alpha}_{mn}(t) + \dot{\delta}_{mn}(t) = -(g - Z_0\Omega^2\cos\Omega t)\,A_{mn}(t)a + \dot{\delta}_{mn}(t) \quad (168)$$

are obtained.

It should be pointed out that $\dot{\alpha}_{oo}(t)$ and $\dot{\beta}_{oo}(t)$ are third order terms, which can be obtained from the dynamical and kinematic conditions of the liquid free surface in terms of $\alpha_{mn}(t)$, $\beta_{mn}(t)$, and $a_{mn}(t)$ and their derivatives. If the values of $a_{mn}(t)$, $\alpha_{mn}(t)$, and $\beta_{mn}(t)$ are determined, then $\dot{\alpha}_{oo}(t)$ and $\dot{\beta}_{oo}(t)$ may be obtained. Therefore the forces and moments of the liquid motion can be obtained by inserting the liquid surface response $A_{mn}(t)$ into the expressions of the forces and moments (equations (160) through (164)).

In the overall stability analysis of a space vehicle, it is advantageous to provide an equivalent mechanical model which would describe the liquid motion. For the case of a longitudinally excited, circular cylindrical container with rigid walls and bottom, an equivalent nonlinear mechanical model has been derived.[22] This model may be modified slightly to describe the liquid motion in a container with an elastic bottom.

CHAPTER VI

DISCUSSION OF THE RESULTS AND CONCLUSIONS

For the numerical evaluation of the analytical results, it has been restricted mainly to a container 400 inches in diameter with an elastic bottom of thickness $\bar{h} = 0.195$ inch. The container is filled with liquid to a height of $H = 2.0a$ or $0.2a$. The responses have been determined for various excitation amplitudes Z_o ($Z_o = 0.001a, 0.005a, 0.01a$). In each case the uncoupled one-half subharmonic and harmonic responses as well as the coupled one-half subharmonic and harmonic responses have been determined. These results correspond to one of the booster tanks of the Saturn V space vehicle. To show the difference of the response of the liquid in a container with rigid bottom ($D \to \infty$), the above cases for such a container have been obtained also and are represented in the figures as dotted lines. Thus the influence of the elasticity of the container bottom can be lucidly demonstrated. In addition, for the comparison of the influence of the various bottom thicknesses upon the liquid responses, the numerical calculation for a container 400 inches in diameter with a bottom thickness of one inch have also been carried out.

To compare the calculated results with experimental and other available analytical results for a rigid bottom,[10,11] the radius of the container is chosen to be $a = 2.86$ inches. The liquid heights are $H = 2a$ or $0.5a$, and the excitation amplitudes are $Z_o = 0.00903a, 0.00451a$.

In the numerical evaluation of this case, it has been restricted to the axisymmetric response. All response curves represent the surface elevation η^* versus the ratio of the forcing frequency $\frac{1}{2}\Omega$ (for one-half subharmonic response) or Ω (for harmonic response) to the fundamental natural circular linearized frequency of the liquid ω_1.

One-half Subharmonic Response

The mean liquid amplitudes η^* for the axisymmetric case are calculated from

$$\eta^* = \frac{1}{2}\left\{\left|\eta\left(r=0, \frac{\Omega t}{2}=\frac{\pi}{2}\right)\right| + \left|\eta\left(r=0, \frac{\Omega t}{2}=\frac{3\pi}{2}\right)\right|\right\} \tag{169}$$

where η is given in equations (149) and (150) for uncoupled and coupled modes, respectively.

In all of the numerical examples, the liquid is water and the elastic bottom is aluminum. A schematic representation of the one-half subharmonic response of the liquid is shown in Figure 3. For a given excitation amplitude Z_0, the upper curve corresponds to the first solution of the liquid response which is the stable solution and can be observed from the experiment. The responses of the liquid in a container of 400 inches diameter with various liquid heights, bottom thicknesses, and excitation amplitudes are shown in Figures 4 through 9 for both uncoupled and coupled (coupled modes) liquid motions. The solid lines express the liquid responses in a container with elastic bottom, while the dotted lines represent the responses of liquid in a container with rigid bottom. There are two response curves for a given excitation amplitude Z_o, one corresponds to the stable solution and the other to the unstable solution. The stable liquid response is obtained by substituting either equation (149) or equation (150) into equation (168) for uncoupled or coupled liquid motion, respectively. For a given excitation amplitude Z_o, the range of the unstable plane surface of the liquid in a container with the elastic bottom decreases somewhat in comparison with the case of a rigid bottom (for example, see Figure 4). Furthermore, for a given liquid height, say $\frac{H}{a} = 2$, the influence of the elastic bottom upon the range of the unstable plane surface and the liquid response are greater as the thickness of the bottom decreases. This may be seen from Figures 4 and 5. With the tank diameter 400 inches and the bottom thickness one inch, for example, the difference of the liquid response between the cases of the elastic bottom and the rigid bottom is almost negligible. But for the same tank diameter with a different bottom thickness, say 0.195 inch, the influence of the elastic bottom upon both the liquid response and the range of the unstable plane surface is more significant when we compare it with the case of the rigid bottom (see Figure 4). In general, for the ratio of the tank bottom thickness to the tank radius $\frac{\bar{h}}{a} \geq \frac{1}{200}$, and the ratio of the liquid height to the tank radius $\frac{H}{a} \geq 1$, the influence of the elastic bottom upon both the liquid response and the range of the unstable plane surface is not large. With the same tank diameter, excitation amplitude, and bottom thickness (say $a = 200$ inches, $\frac{Z_o}{a} = 0.005$, $\bar{h} = 1$ inch), the elastic effect is more significant as the liquid height H is decreased. This can be seen from Figures 5 and 6. The excitation amplitude Z_o is also a factor which affects the liquid response as well as the range of the unstable plane surface. In general, the stable response of a liquid in a container with an elastic bottom is smaller than that of the rigid bottom. Finally, it should be pointed out that the responses for both uncoupled and coupled liquid motion exhibit softer nonlinear characteristics,[23] the coupled motion is softer than the uncoupled liquid motion. But with the parameter $\frac{H}{a}$ decreasing to a certain number, say $\frac{H}{a} = 0.2$, the response of the coupled liquid motion exhibits hard nonlinear characteristics (see Figure 9).

Harmonic Response

The harmonic response means that the period of the response of the liquid motion is exactly the same as the period of the tank excitation. The harmonic response has been observed experimentally. However, the liquid amplitude was found to be smaller than that of the one-half subharmonic liquid motion. In comparison with experimental results (for the case of rigid bottom), it was found that the theoretical prediction of the range of the unstable plane surface is much smaller than that observed experimentally. Furthermore, the theoretical prediction of the liquid response is much larger than that obtained from experimental work.[10]

The harmonic response of the liquid in a container with an elastic bottom is quite interesting. Because of the fact that the nonhomogeneous term with harmonic excitation appears on the right hand side of equation (81) for the uncoupled liquid motion, an approximate steady solution of equation (81) may be obtained as shown in equation (92). Similarly, because the nonhomogeneous terms with harmonic excitation appear on the right hand sides of equations (112) and (113) for the coupled liquid motion, an approximate steady solution may also be obtained as shown in equations (128) and (129).

The mean liquid amplitudes η^* for the axisymmetric case were calculated from

$$\eta^* = \frac{1}{2}\left\{\left|\eta(r=0, \Omega t=0)\right| + \left|\eta(r=0, \Omega t=\pi)\right|\right\} \qquad (170)$$

where η is given in equations (151) and (152) for uncoupled and coupled modes, respectively.

The harmonic responses of the liquid in a container of 400 inches diameter with various liquid heights, bottom thicknesses, and excitation amplitudes are shown in Figures 10 through 15 for both uncoupled and coupled liquid motions. The solid lines represent the responses of the liquid in a container with an elastic bottom. The dotted lines represent the responses of the liquid in a container with a rigid bottom. In the case of an elastic bottom, it can be seen that the amplitudes of the harmonic liquid motion always exist throughout the forcing frequency range (for example, see Figure 10) even though the liquid response might be a small value only in some range. This means, in the case of the elastic bottom, that there is no plane surface in the range of forcing frequency at all in contrast to the one-half subharmonic response in which a plane surface exists in certain ranges of the forcing frequencies (see Figure 3).

For a given liquid height (say $\frac{H}{a} = 2.0$), the influence of the elastic bottom upon the liquid response is more significant when the thickness of the bottom decreases (for example, when $\bar{h} = 0.195$ inch and $\bar{h} = 1$ inch, see Figures 10 and 11). For a given tank diameter with the same bottom thickness and the excitation amplitude Z_0, the elastic effect upon the liquid response is more obvious as the liquid height decreases (for example, when $\frac{H}{a} = 2$, and 0.2, see Figures 11 and 12). The excitation amplitude is also a factor which affects the liquid response. Both uncoupled and coupled harmonic responses are exhibiting softer nonlinear characteristics. The coupled motion is softer than the uncoupled liquid motion. In general, the harmonic response is smaller than the one-half subharmonic response (by comparison of Figures 10 through 15 with Figures 4 through 9).

Figures 16 through 23 show the one-half subharmonic and harmonic responses of the liquid in a container of 2.86 inches diameter with both rigid bottom and elastic bottom thickness of $\overline{h} = 0.003$ inch. The liquid heights are chosen as $\frac{H}{a} = 2.0$ or 0.5 while the excitation amplitudes are $\frac{Z_o}{a} = 0.00451$ and 0.00903. The experimental data are for the response of the liquid in a container with a rigid bottom. For the case of the one-half subharmonic response, the coupled liquid motion gives a good comparison with the experimental results for small η^*.

In summary, the liquid surface response in a circular cylindrical container when subjected to a longitudinal excitation is very complex in general. In the lower range of frequencies, it was found experimentally that the first antixymmetric and axisymmetric modes could be somewhat isolated. From the design of the space vehicle point of view, the liquid motion in the lower frequency range is more important. Therefore, in this dissertation, it has been restricted to the analysis of the limit of the lower frequency range. A third order theory was presented in order to minimize the difficulties in the computation. More accurate results could be obtained by extending the analysis up to the fifth or higher orders. With the consideration of the effect of the elastic bottom, first, the liquid motion was considered to be composed of one primary mode only (first axisymmetric mode) and no coupling was allowed; second, the analysis has been extended to the coupled liquid motion in which the first axisymmetric mode was considered as a primary mode, while the second axisymmetric mode was assumed to be a secondary mode. Both one-half subharmonic and harmonic liquid responses were investigated in more detail with respect to the uncoupled as well as the coupled liquid motions.

Finally, it is recommended that the same analysis be used to investigate the liquid motion of the antisymmetric modes. Additional experimental data for the case of the elastic bottom are needed in order to compare with theoretical results.

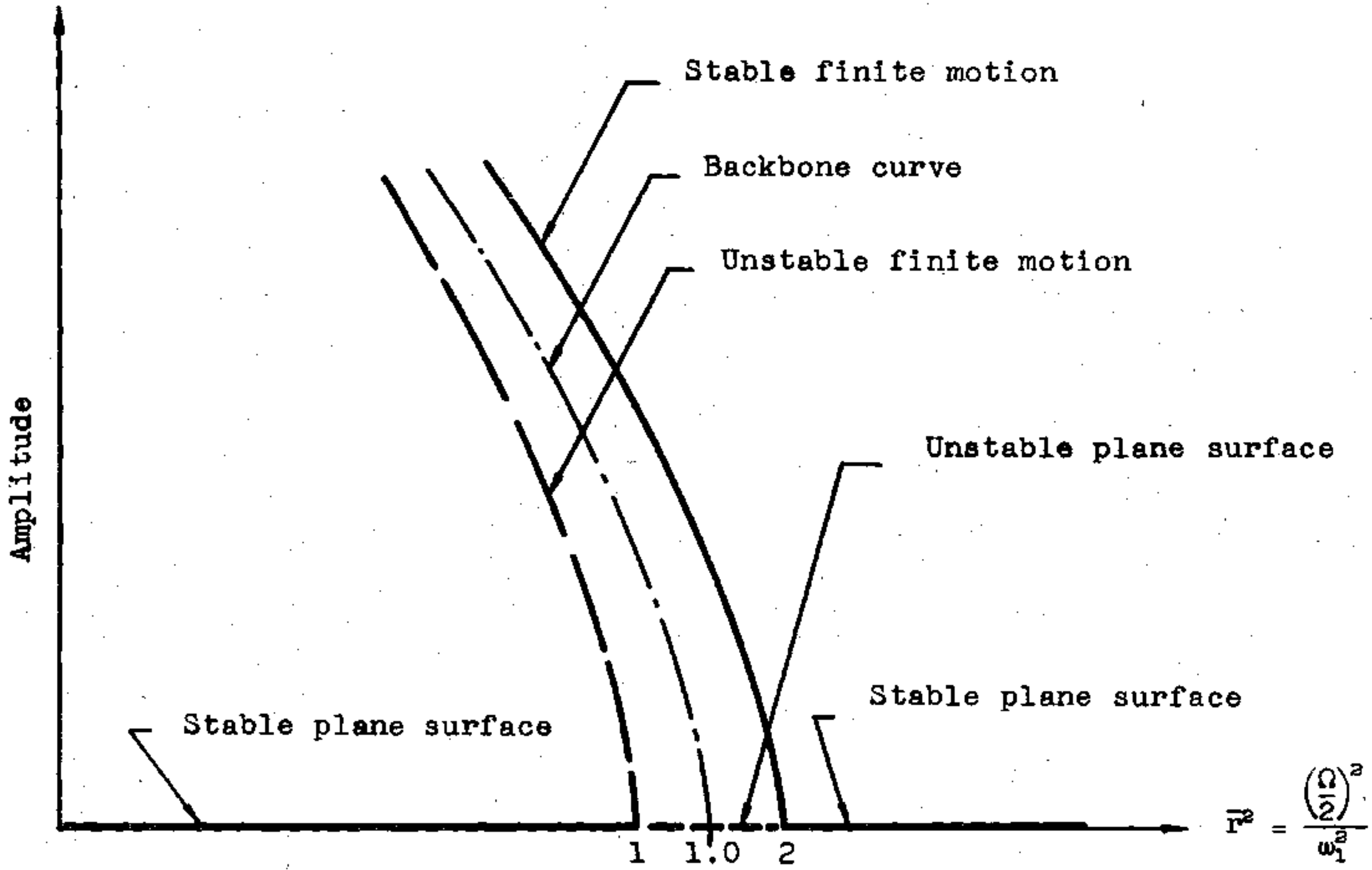
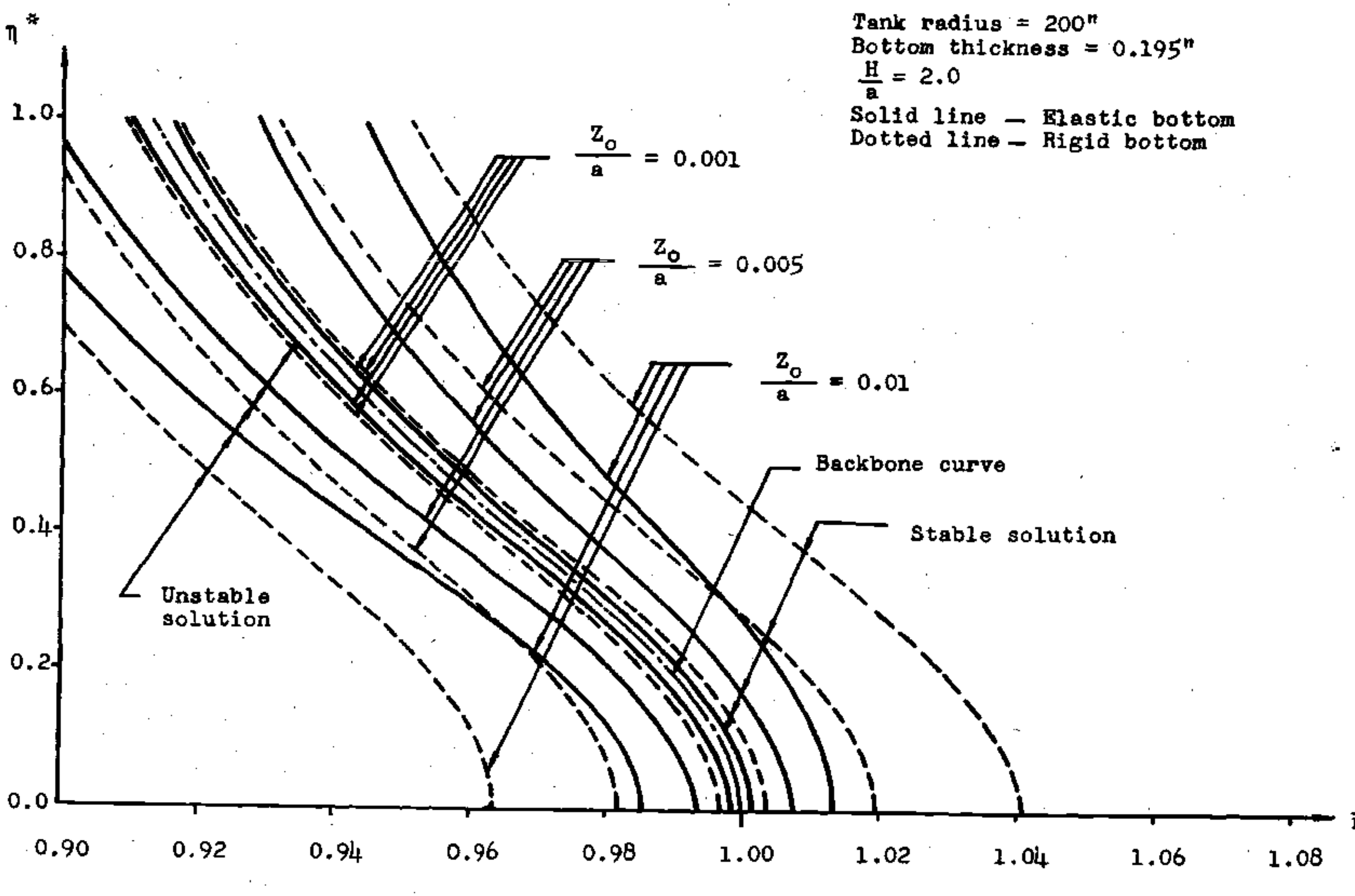


Figure 3. Schematic Representation of the One-half Subharmonic Response



$\bar{r} = \frac{\Omega}{2}/\omega_1$

Figure 4. Uncoupled One-half Subharmonic Response

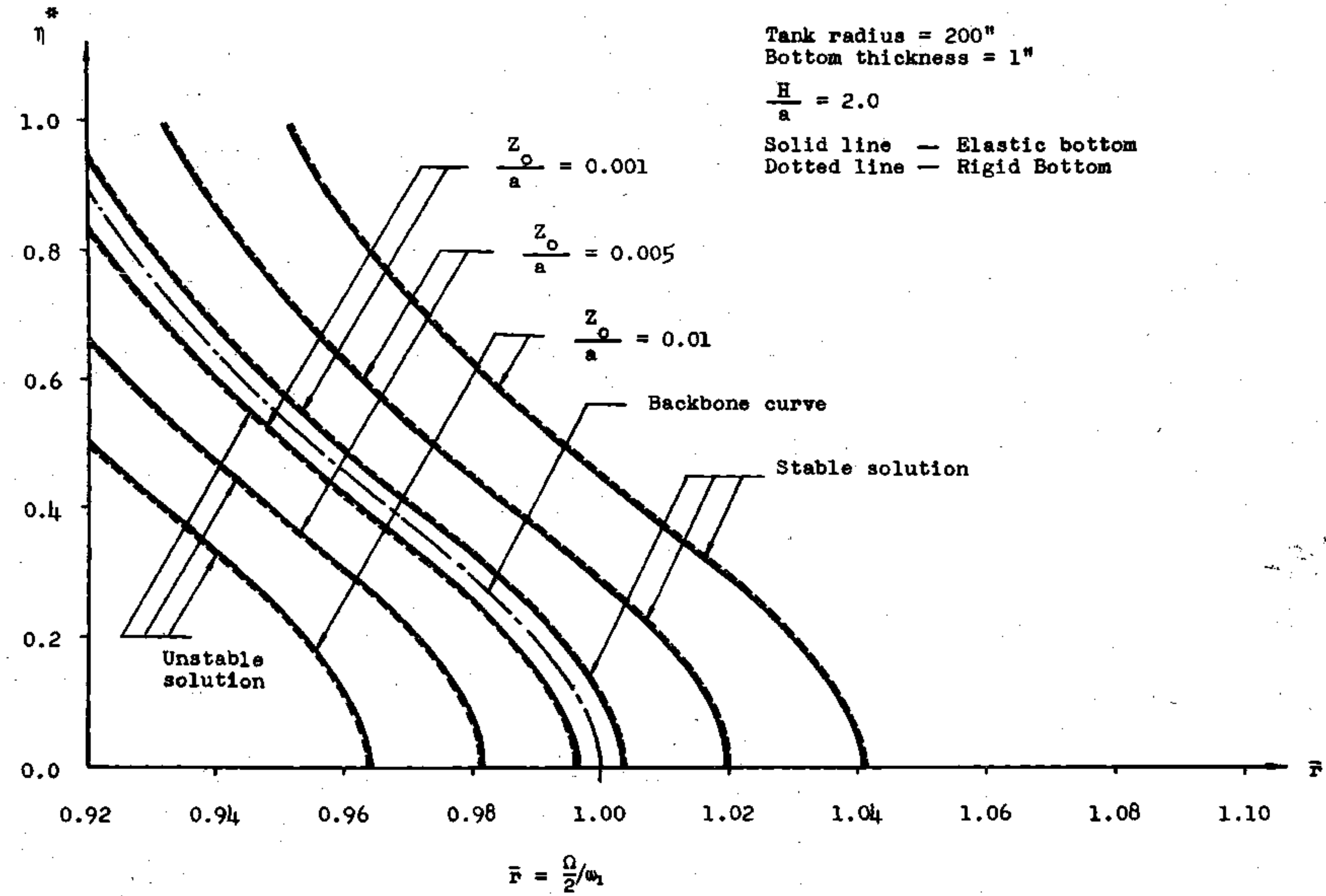


Figure 5. Uncoupled One-half Subharmonic Response

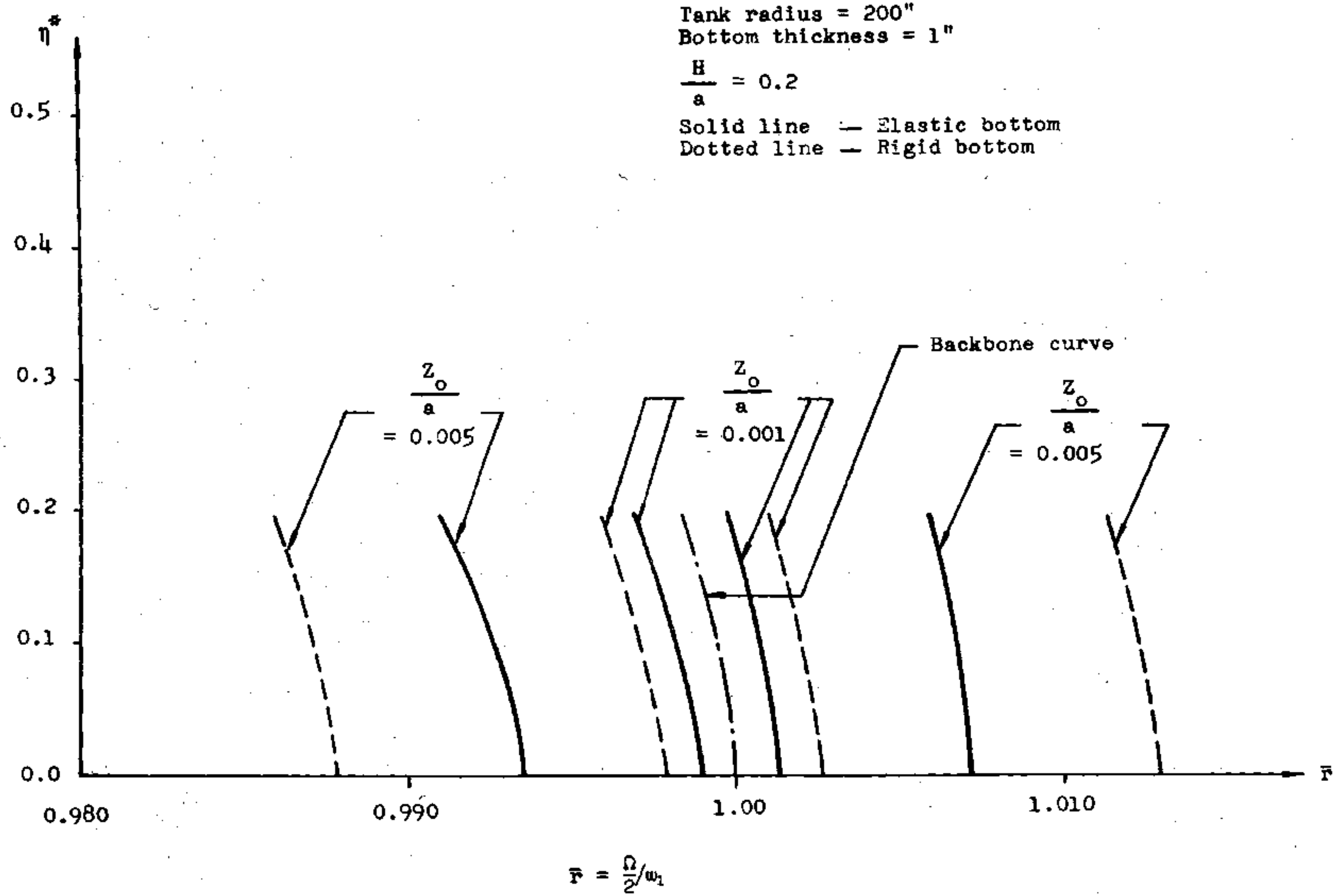


Figure 6. Uncoupled One-half Subharmonic Response

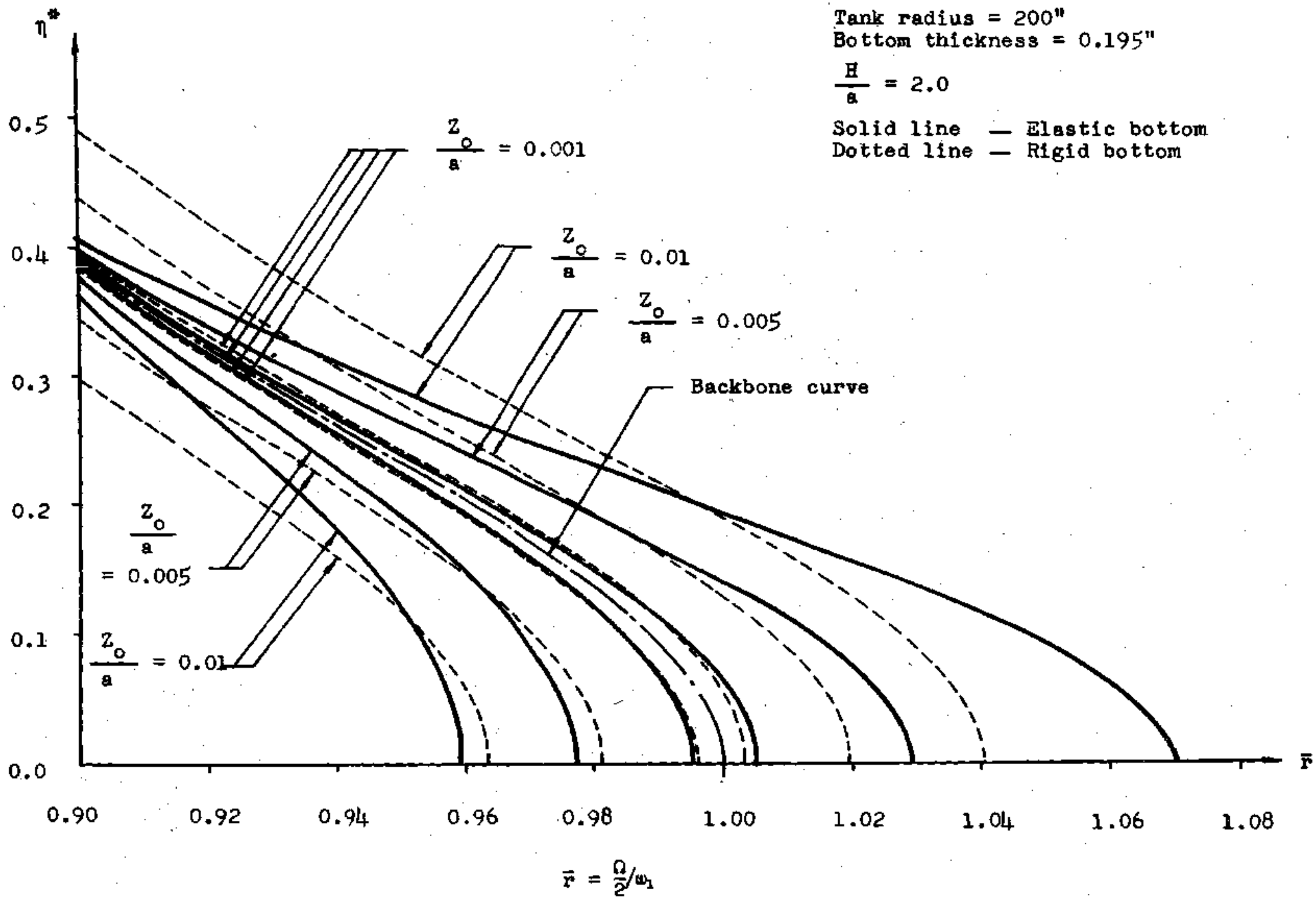


Figure 7. Coupled One-half Subharmonic Response

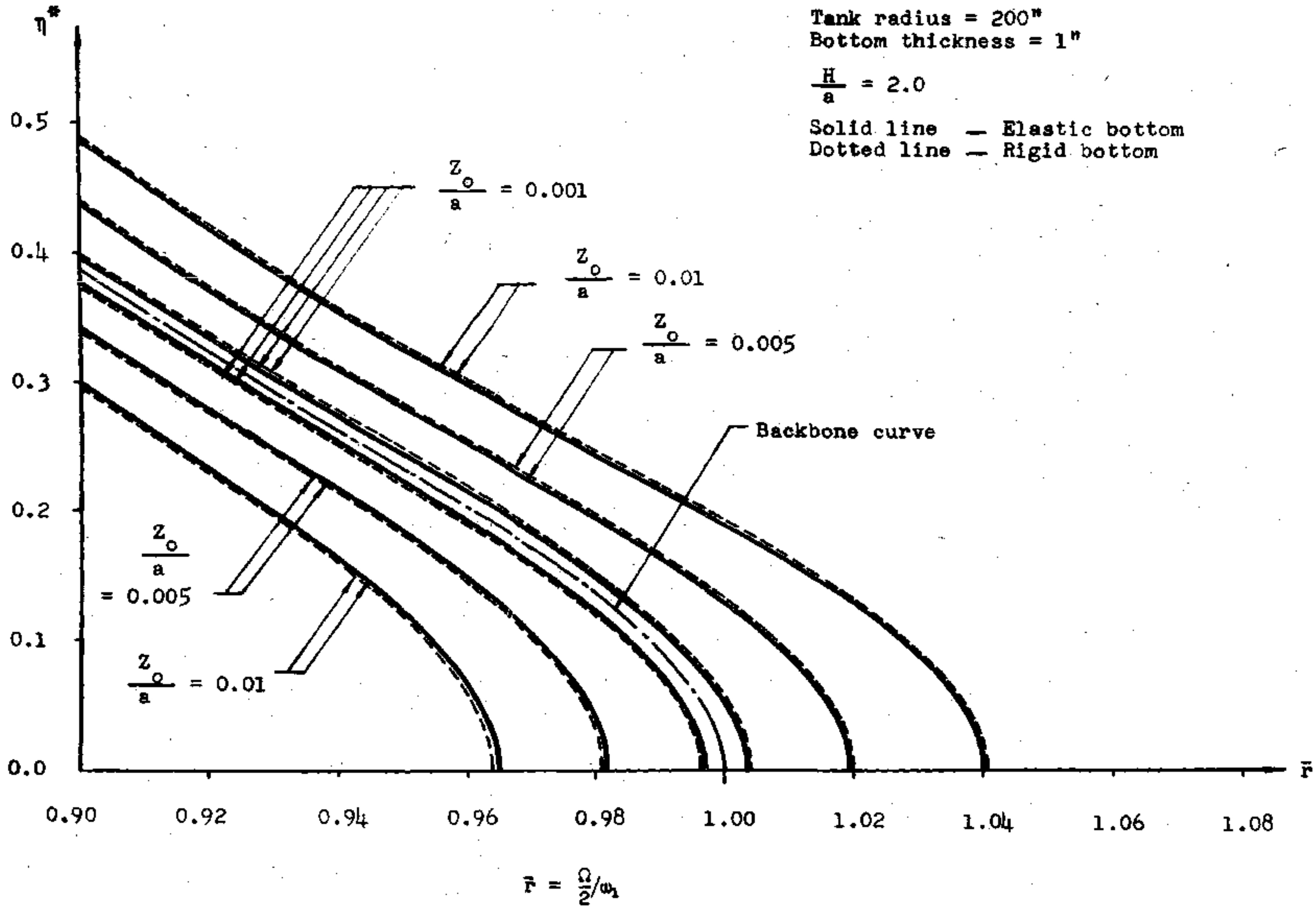


Figure 8. Coupled One-half Subharmonic Response

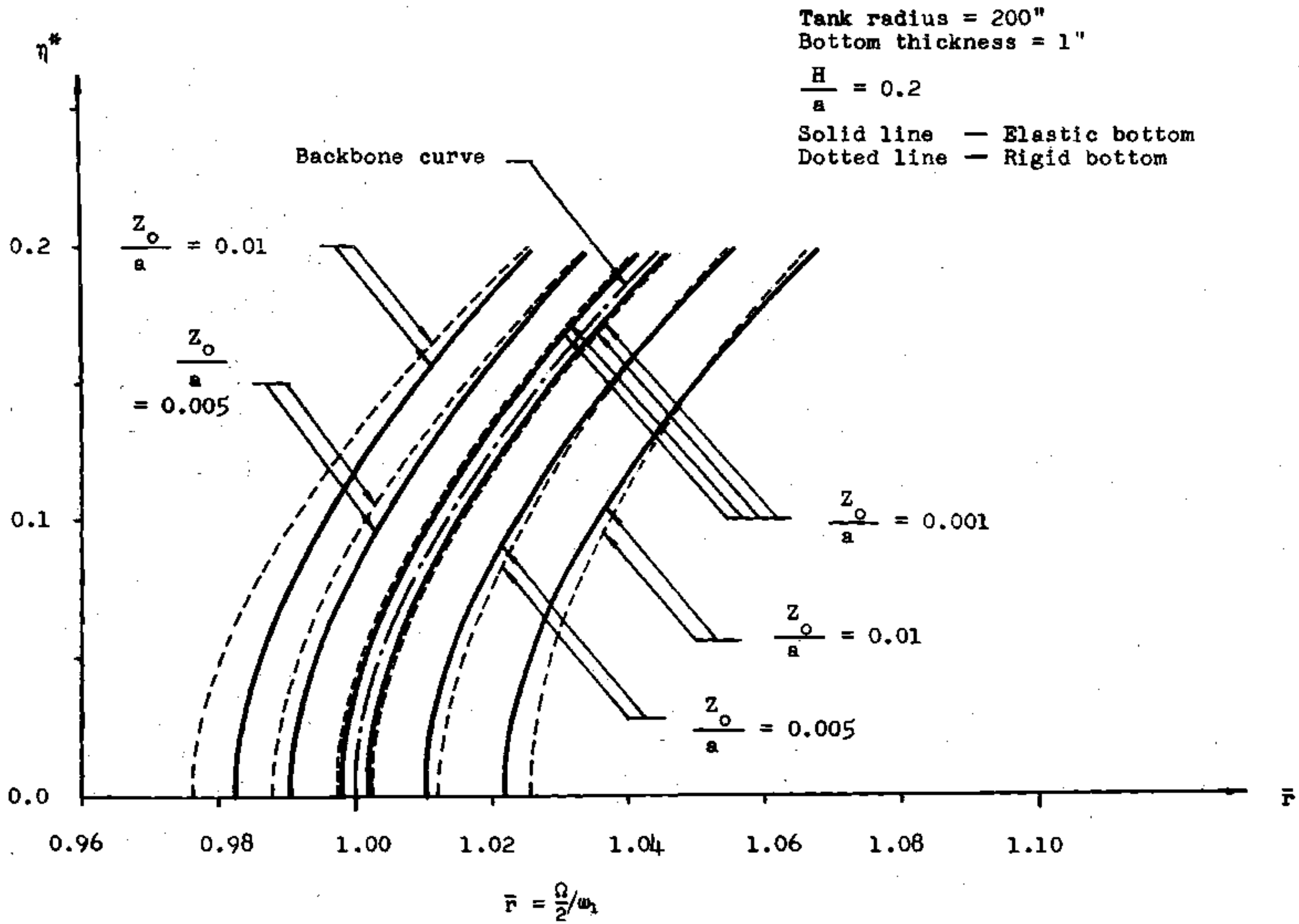


Figure 9. Coupled One-half Subharmonic Response

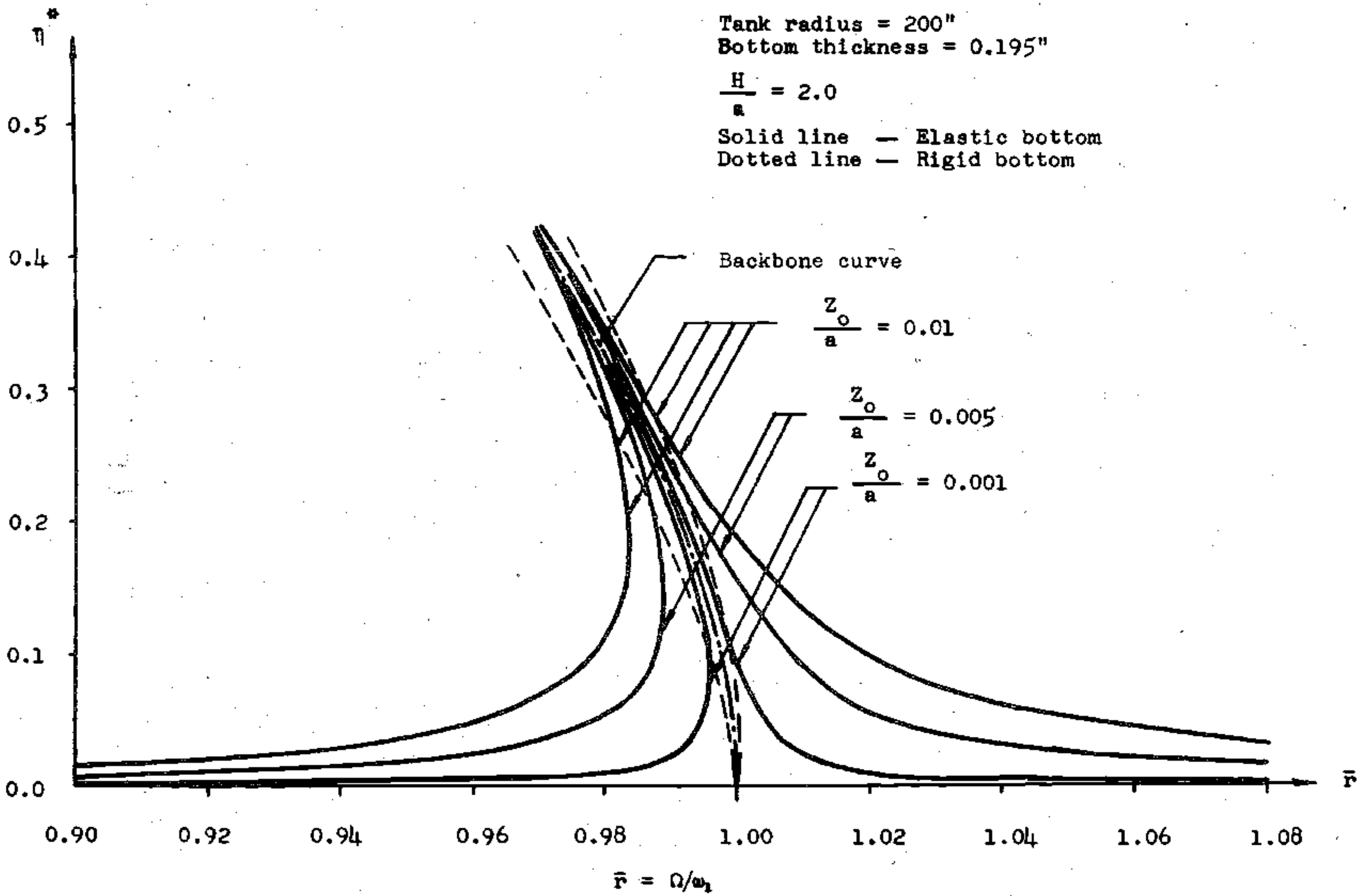


Figure 10. Uncoupled Harmonic Response

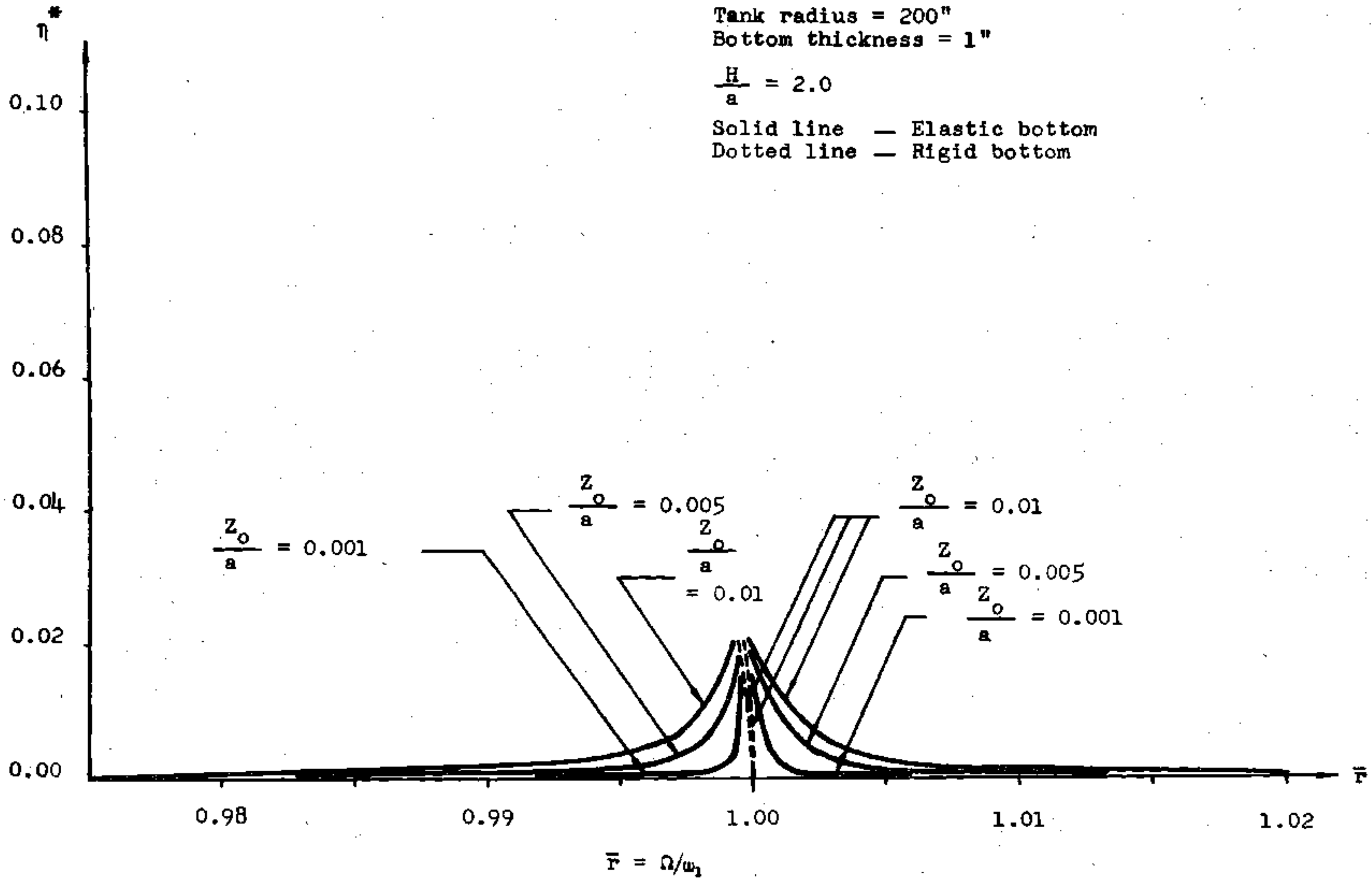


Figure 11. Uncoupled Harmonic Response

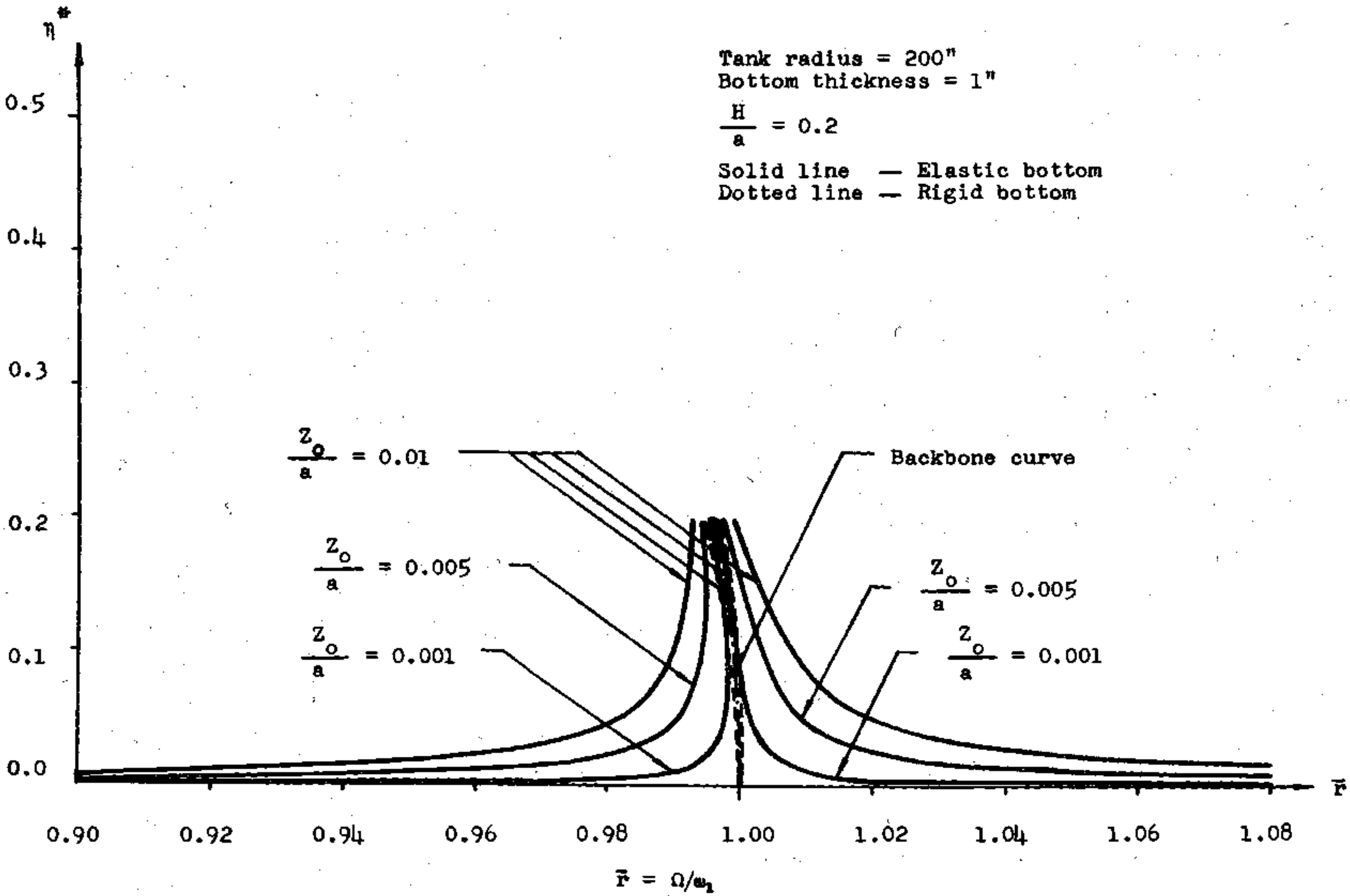


Figure 12. Uncoupled Harmonic Response

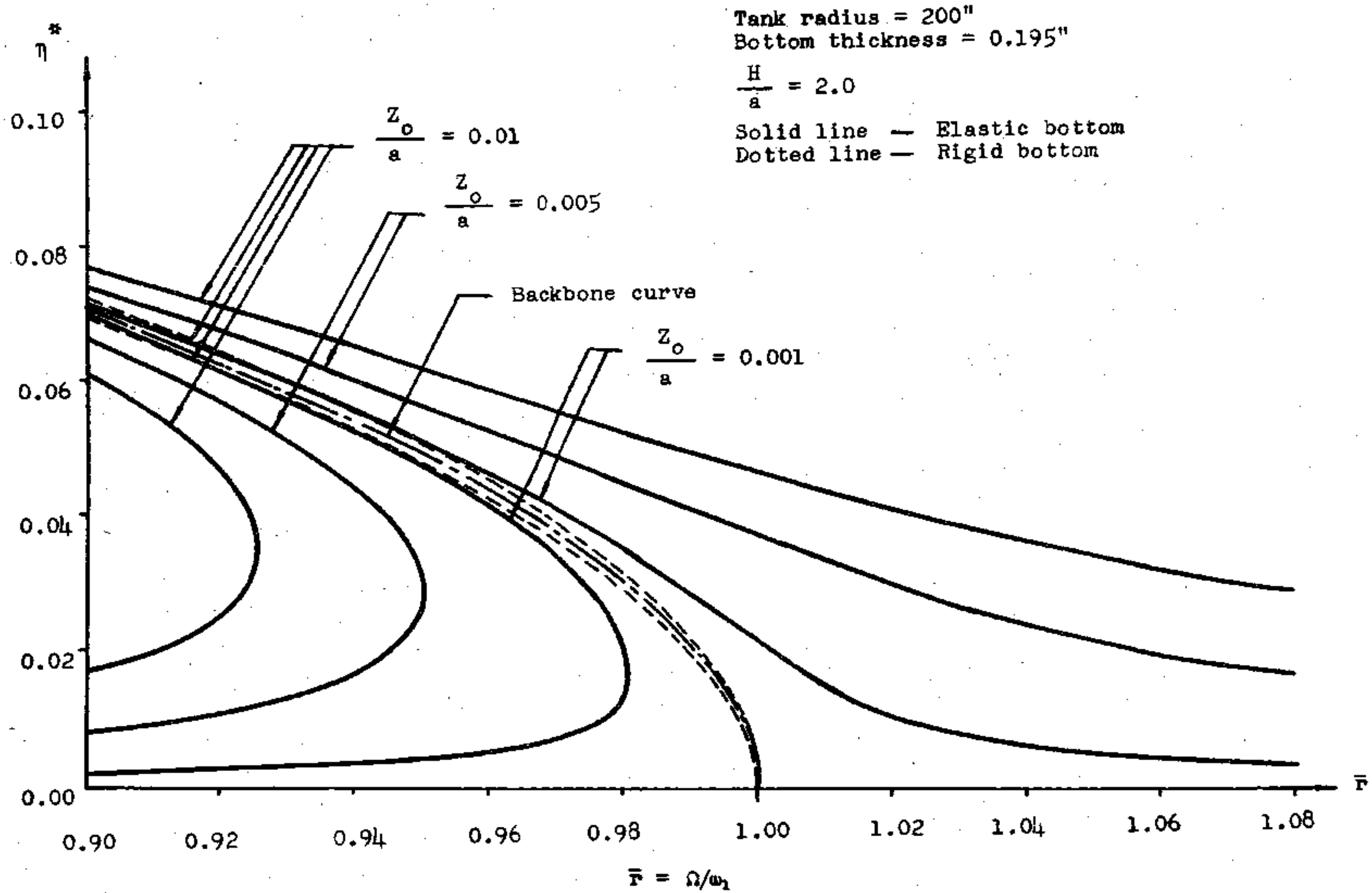


Figure 13. Coupled Harmonic Response

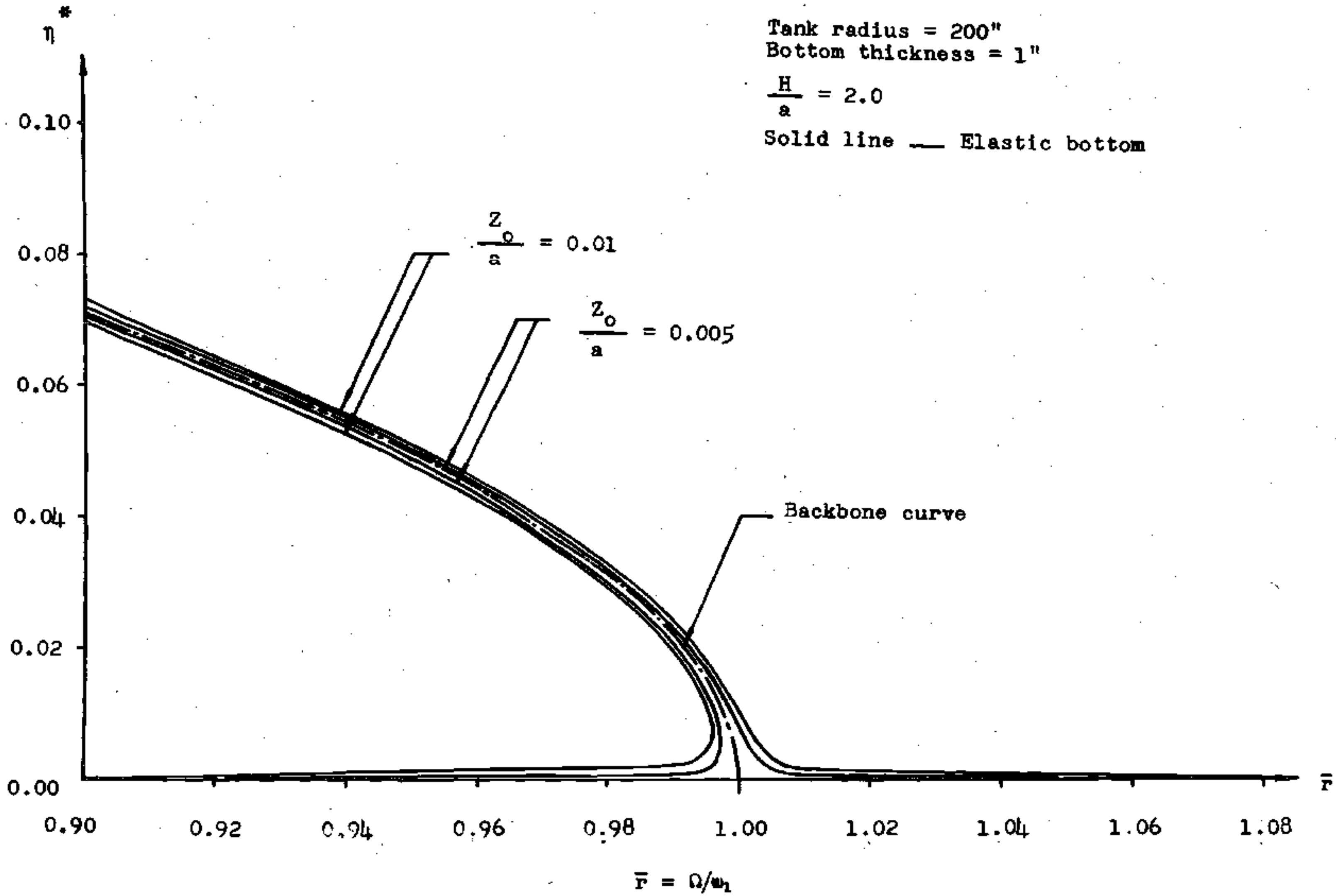


Figure 14. Coupled Harmonic Response

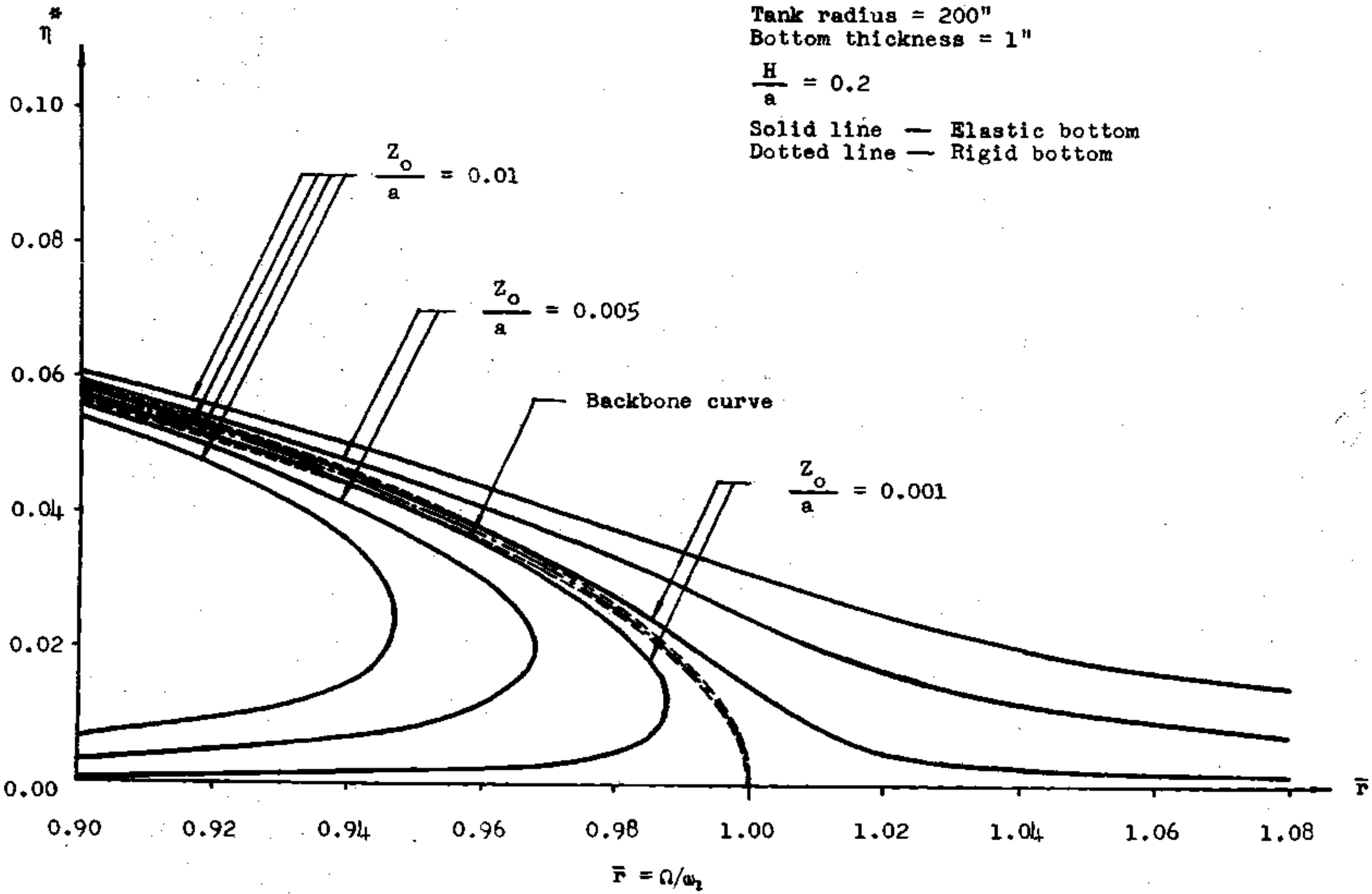


Figure 15. Coupled Harmonic Response

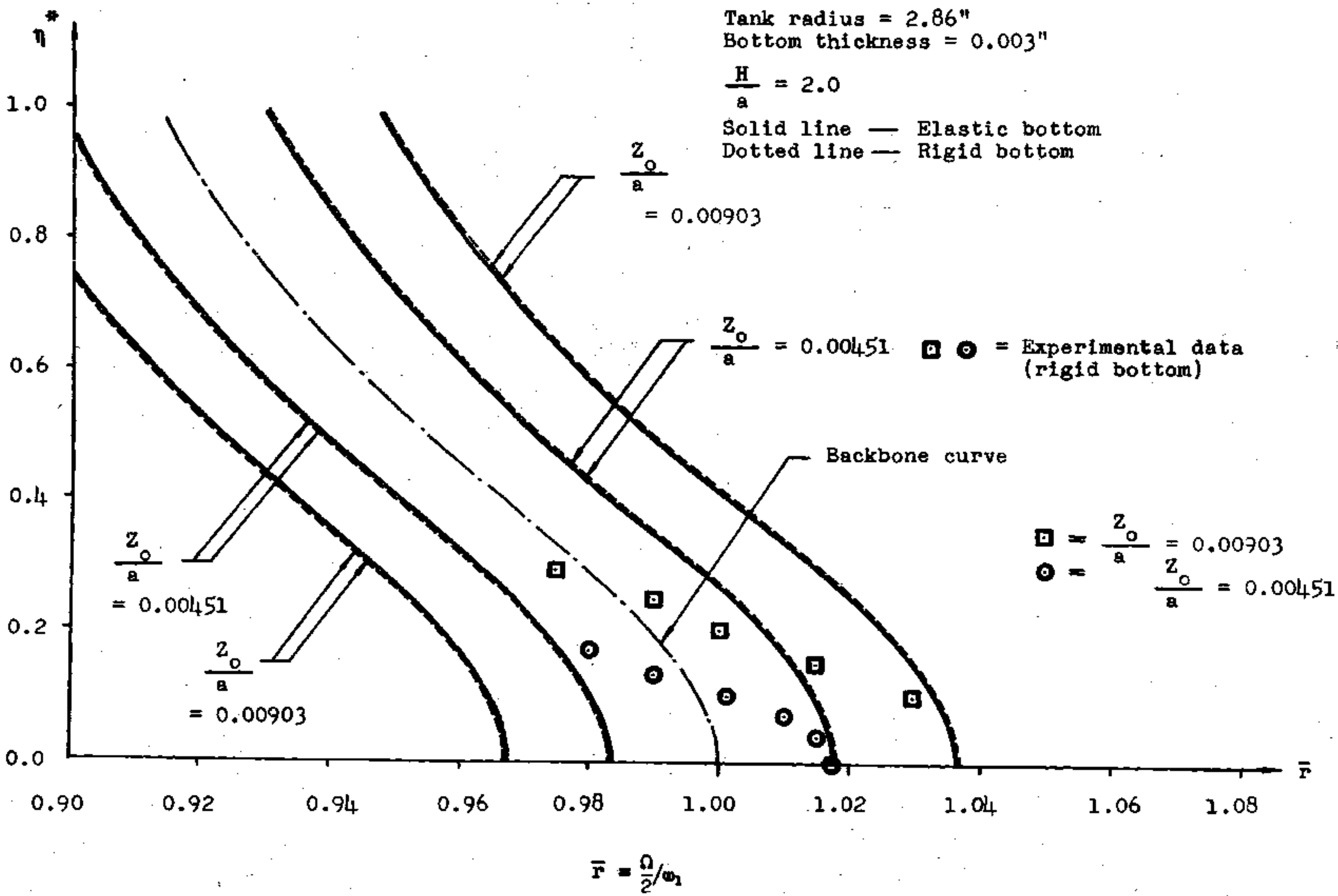
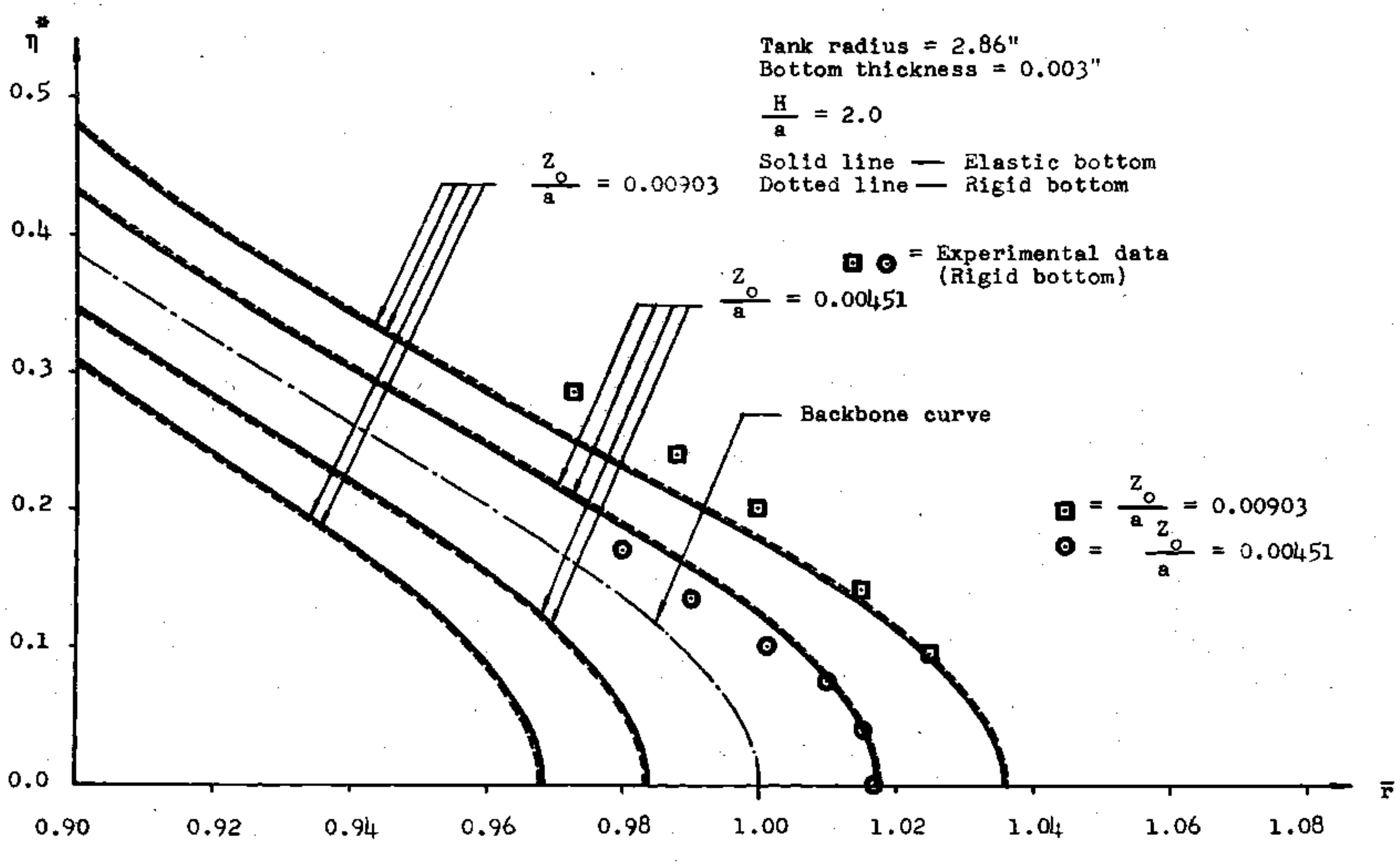
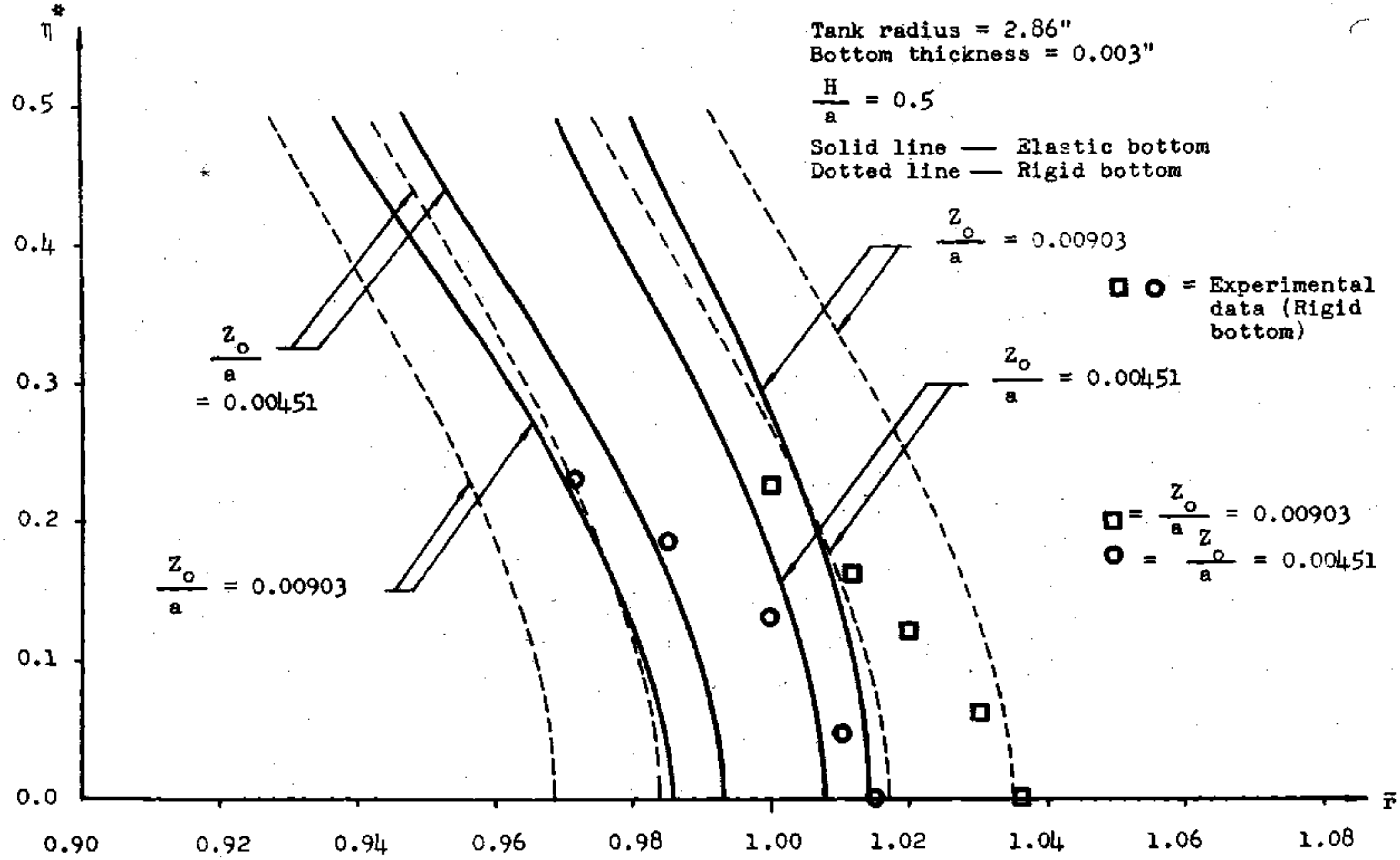


Figure 16. Uncoupled One-half Subharmonic Response



$\bar{r} = \frac{\Omega}{2}/\omega_1$

Figure 17. Coupled One-half Subharmonic Response



$\bar{r} = \frac{\Omega}{2}/\omega_1$

Figure 18. Uncoupled One-half Subharmonic Response

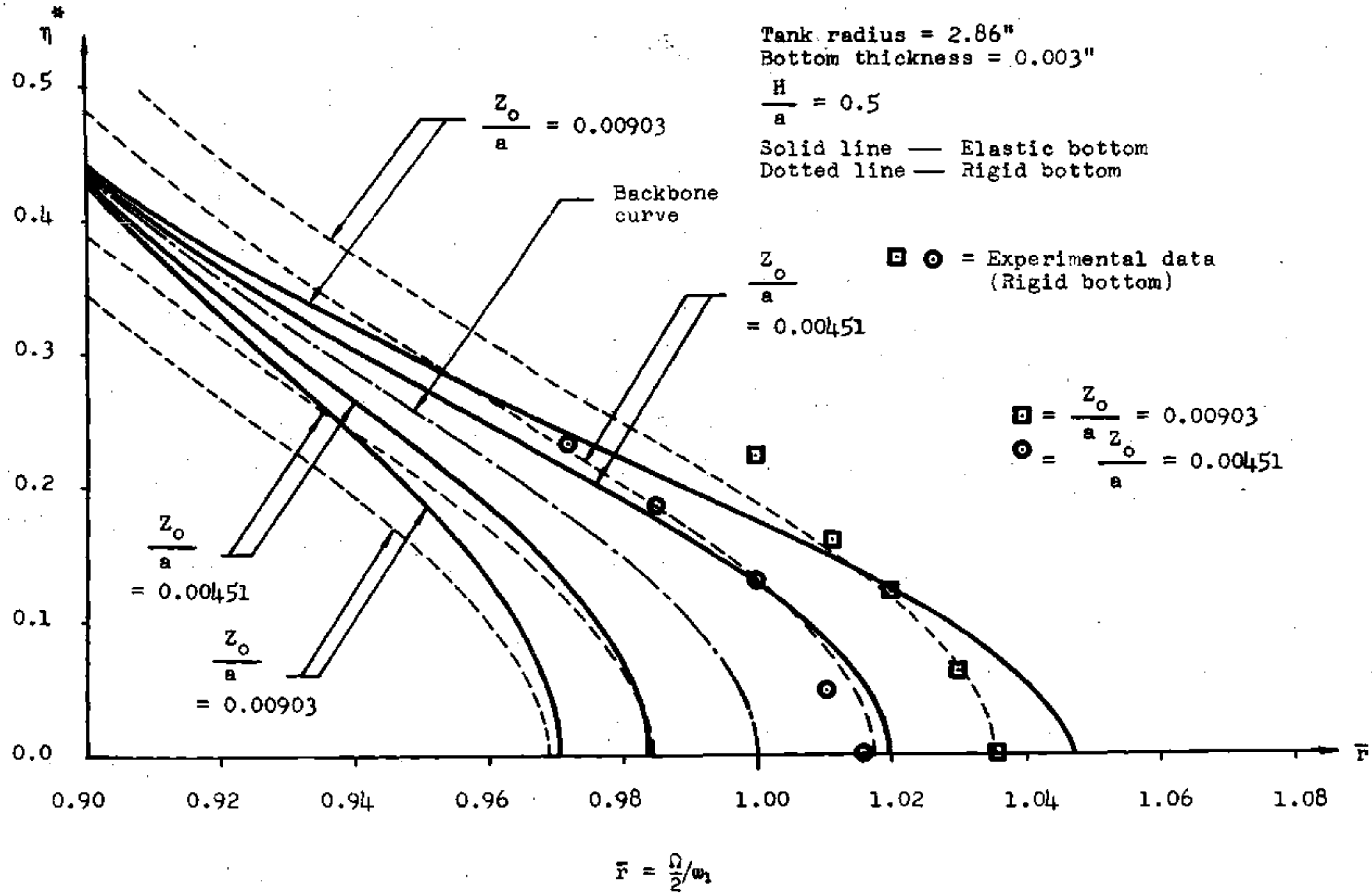


Figure 19. Coupled One-half Subharmonic Response

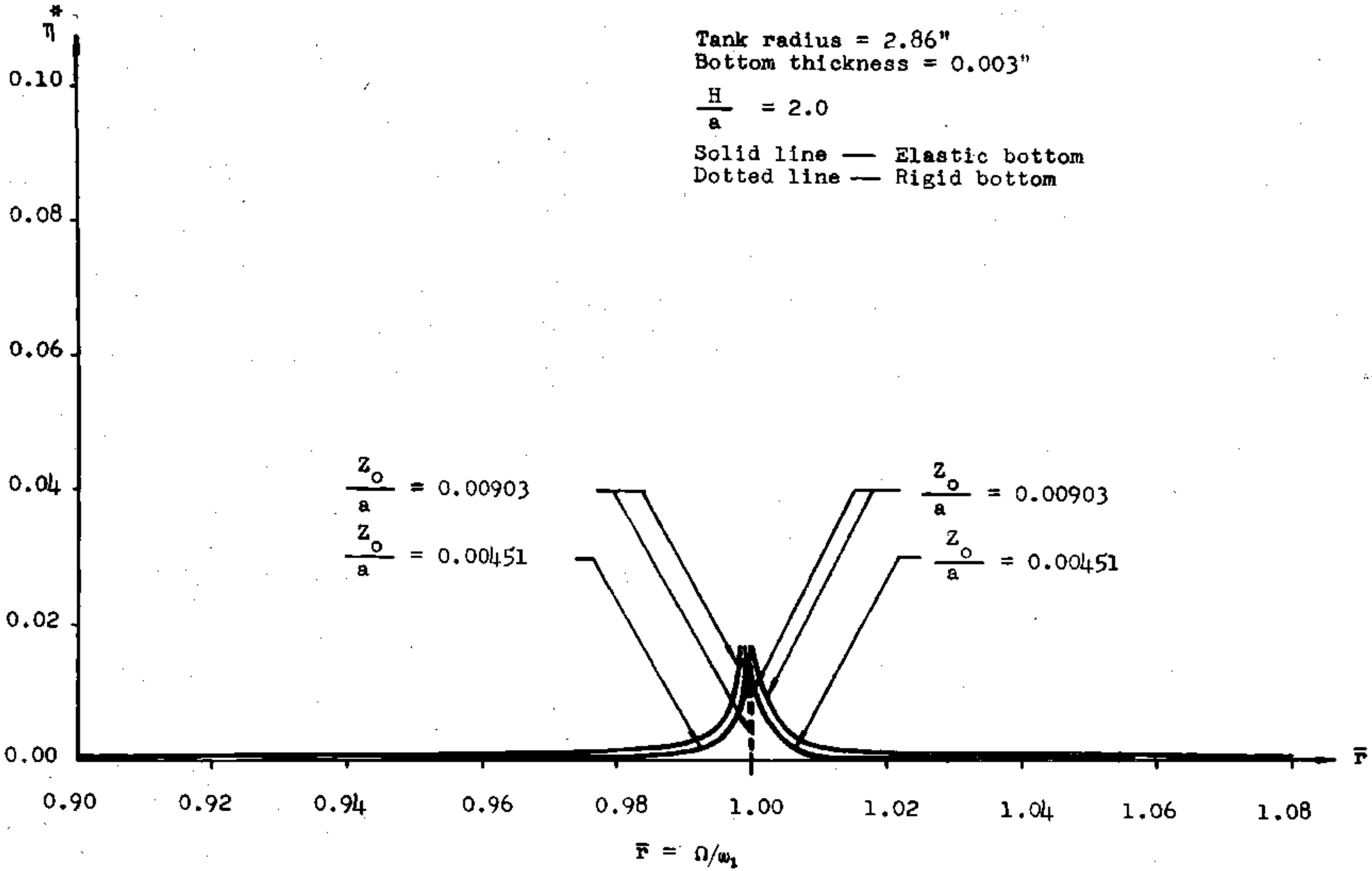


Figure 20. Uncoupled Harmonic Response

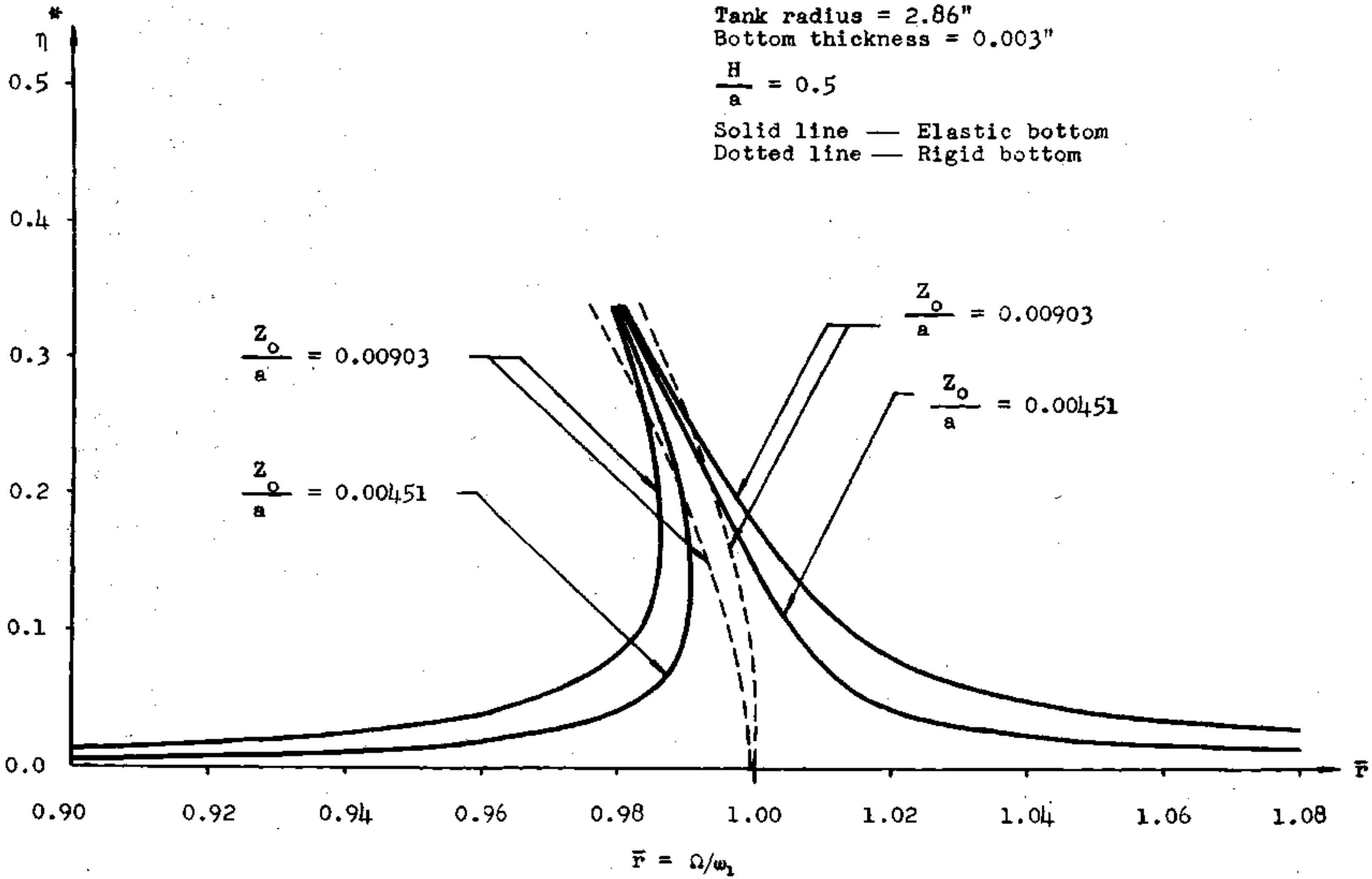


Figure 21. Uncoupled Harmonic Response

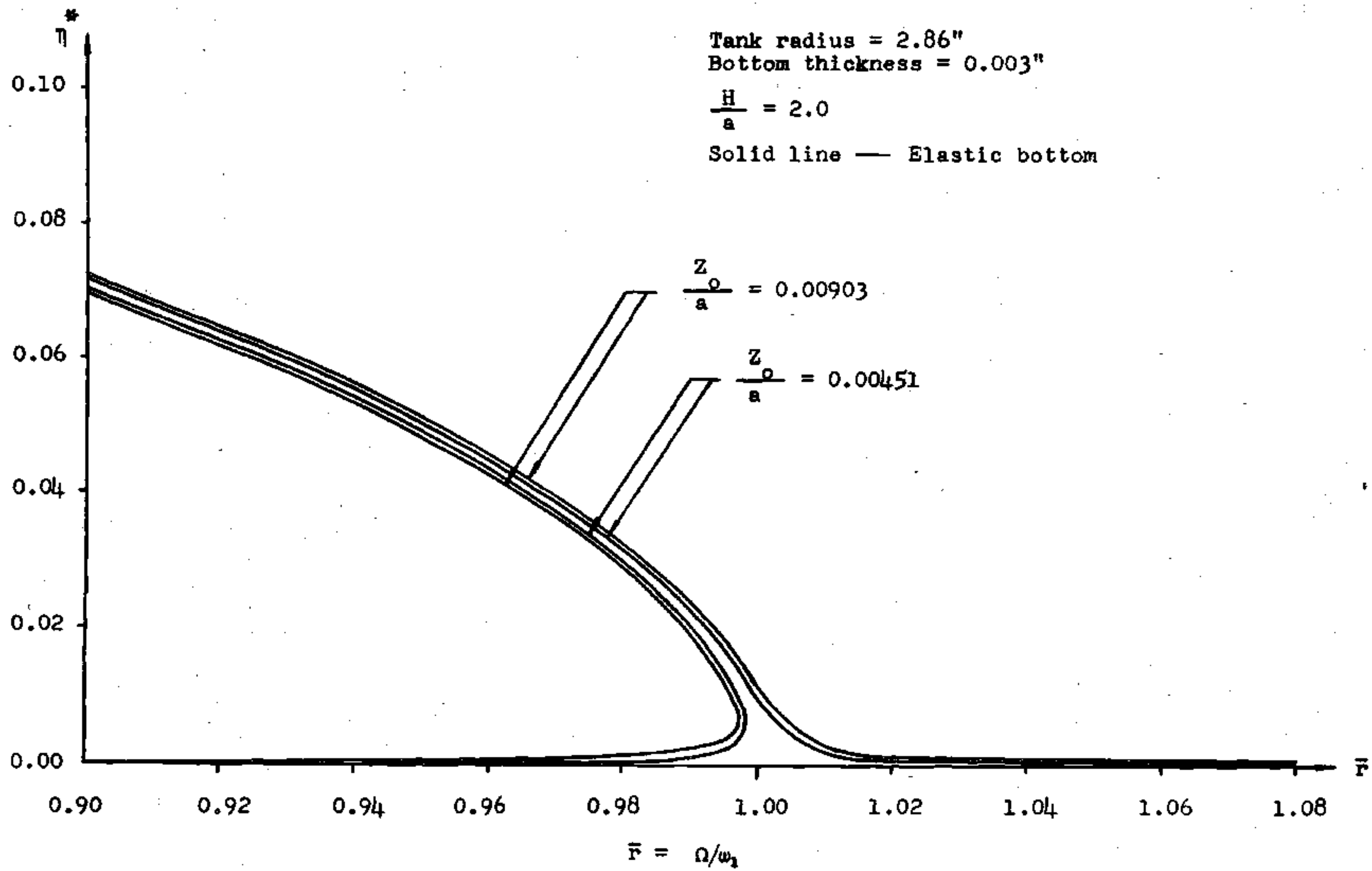


Figure 22. Coupled Harmonic Response

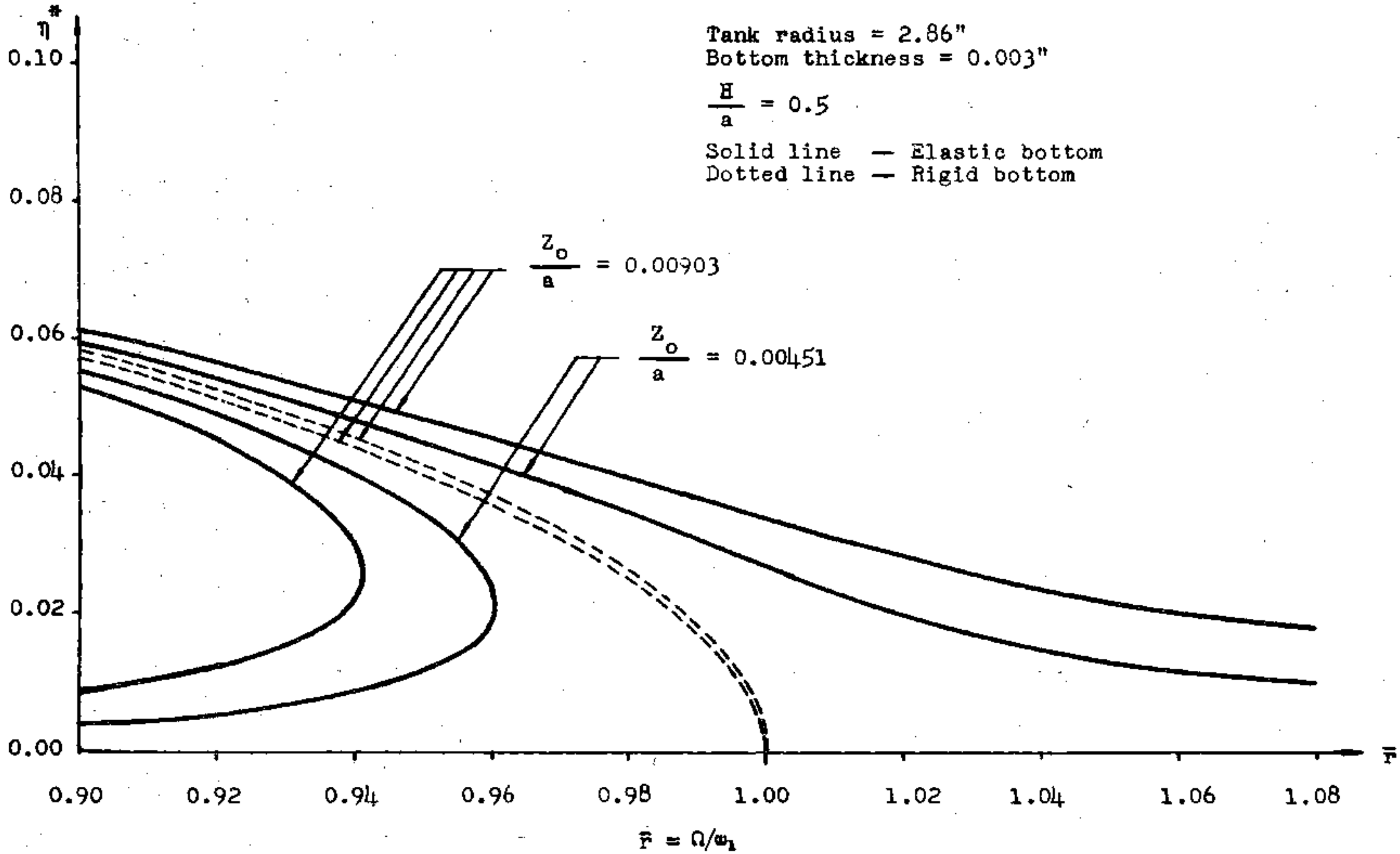


Figure 23. Coupled Harmonic Response

APPENDIX A

MATHIEU-HILL EQUATION

The Hill equation may be written as

$$\frac{d^2f}{dt^2} + (\delta - 2\epsilon\psi(t))f = 0 \tag{A-1}$$

where δ and ϵ are parameters, $\psi(t)$ is a periodic function with period T.

If we take $\psi(t) = \cos t$ in (A-1), the result is a special case of Hill's equation called the Mathieu equation:

$$\frac{d^2f}{dt^2} + (\delta - 2\epsilon \cos t)f = 0 \tag{A-2}$$

Any solution of the Mathieu equation will be known as a Mathieu function. Depending on the parameters ϵ and δ the solution of the equation will be bounded or unbounded. In the parameter plane of ϵ and δ, the regions of unboundedly increasing solutions are separated from the region of stability by the periodic solution with period 2π and 4π. The curves denoted by a_i and b_i, as shown in Figure 24, are given by[24]

$$a_1 : \quad \delta = \frac{1}{4} + \epsilon - \frac{\epsilon^2}{2} - \frac{\epsilon^3}{4} + O(\epsilon^4)$$

$$b_1 : \quad \delta = \frac{1}{4} - \epsilon - \frac{\epsilon^2}{2} + \frac{\epsilon^3}{4} + O(\epsilon^4)$$

$$a_2 : \quad \delta = 1 + \frac{5\epsilon^2}{3} + O(\epsilon^4)$$

$$b_2 : \quad \delta = 1 - \frac{\epsilon^2}{3} + O(\epsilon^4)$$

$$a_3 : \quad \delta = \tfrac{9}{4} + \tfrac{1}{4}\epsilon^2 + \tfrac{1}{4}\epsilon^3 + O(\epsilon^4)$$

$$b_3 : \quad \delta = \tfrac{9}{4} + \tfrac{1}{4}\epsilon^2 - \tfrac{1}{4}\epsilon^3 + O(\epsilon^4)$$

$$a_n : \quad \delta = \tfrac{1}{4}n^2 + \frac{2\epsilon^2}{(n^2-1)} + O(\epsilon^4)$$

$$b_n : \quad \delta = \tfrac{1}{4}n^2 + \frac{2\epsilon^2}{(n^2-1)} + O(\epsilon^4)$$

The curves of a_n and b_n with the same subscript number bound the region of instability, i.e., the cross-hatched regions as shown in Figure 24. The curves of b_n and a_{n+1} for n=1, 2, 3 ... bound the region of stability. Since δ is always positive in this analysis, only the first quadrant of the parameter plane is shown in the figure. When $\epsilon=0$, the curves of a_n and b_n coincide at the points $\delta=\frac{1}{4}n^2$ for n=1, 2, 3, ... along the δ-axis. The regions of instability are denoted by Roman numerals with the number of the region increasing as δ increases.

Some properties of the Mathieu function can be summarized as follows.

(1) For values of the parameters which correspond to points on the curves with odd subscripts, the Mathieu function is a periodic function with the period of 4π, i.e., twice that of the forcing function. The form of the Mathieu function is given by

$$f = \sum_{k=1,3,5,\cdots}^{\infty} (A_k \cos \tfrac{1}{2}kt + B_k \sin \tfrac{1}{2}kt) \tag{A-3}$$

Here k is an odd number, A_k and B_k are constants.

(2) For values of the parameters which correspond to points on the curves with even subscripts, the Mathieu function is a periodic function with the period of 2π. The form of the Mathieu function is given by

$$f = A_o + \sum_{k=2,4,6,\cdots}^{\infty} (A_k \cos \tfrac{1}{2}kt + B_k \sin \tfrac{1}{2}kt) \qquad \text{(A-4)}$$

with A_o, A_k, and B_k as constants, and k as an even number.

(3) For values of the parameters which correspond to points in the regions of instability with odd numbers, i.e., the regions bound by curves with odd subscripts, the Mathieu function grows exponentially. The form of the Mathieu function is given by

$$f = e^{\alpha t}\Phi(t) = e^{\alpha t} \sum_{k=1,3,5,\cdots}^{\infty} (A_k \cos \tfrac{1}{2}kt + B_k \sin \tfrac{1}{2}kt) \quad \text{(A-5)}$$

where α is a constant, $0 < \alpha < 1$, and $\Phi(t)$ is a periodic function with the period of 4π. If the unstable point lies in the m^{th} region, the dominating term is $A_m \cos \frac{1}{2}mt + B_m \sin \frac{1}{2}mt$.

(4) For values of the parameters which correspond to points in the regions of instability with even numbers, i.e., the regions bound by curves with even subscripts. The form of the Mathieu function is given by

$$f = e^{\alpha t}\Phi(t) = e^{\alpha t} \sum_{k=0,2,4,\cdots}^{\infty} (A_k \cos \tfrac{1}{2}kt + B_k \sin \tfrac{1}{2}kt) \text{(A-6)}$$

where $\Phi(t)$ is a periodic function with the period of 2π. If the unstable point lies in the m^{th} region, the dominating term is $A_m \cos \frac{1}{2}mt + B_m \sin \frac{1}{2}mt$.

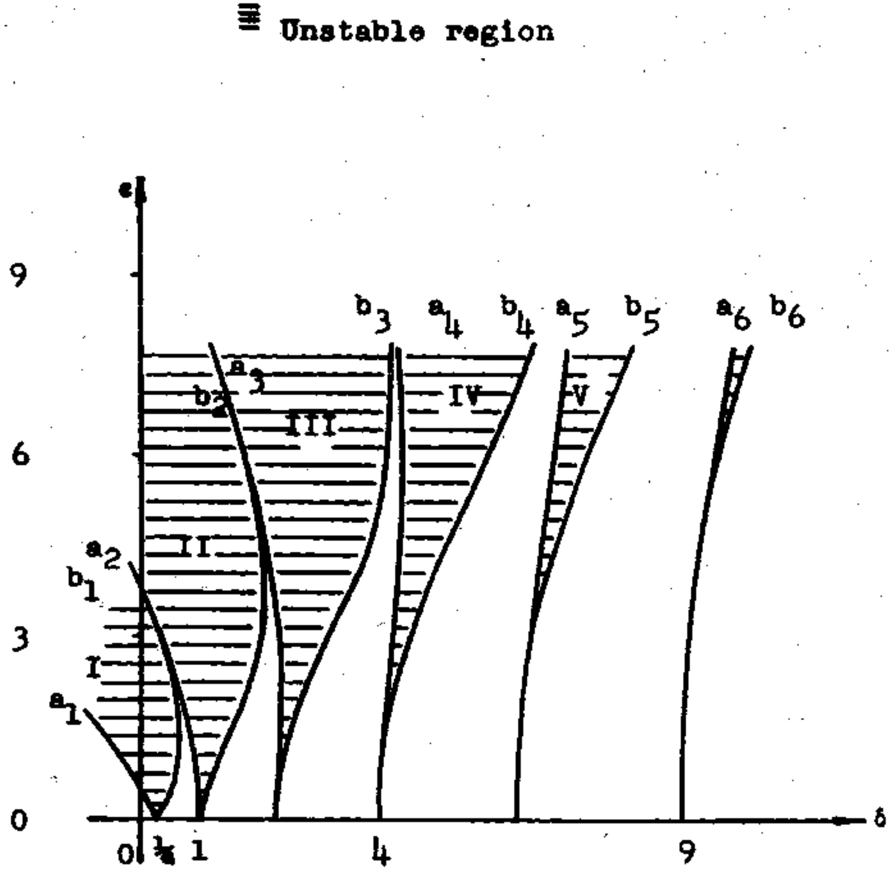


Figure 24. Parametric Plane for Mathieu's Equation

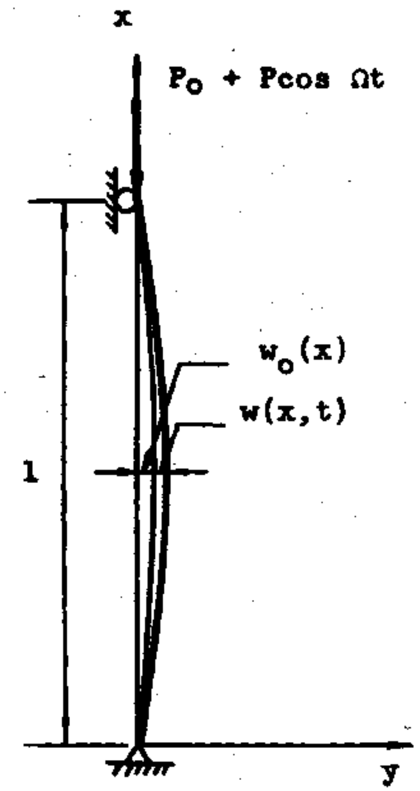


Figure 25. Longitudinally Excited Pinned-End Beam with Its Initial Curvature

APPENDIX B

VARIOUS RELATIONSHIPS

For the sake of clarity, various relationships which have been represented in the previous chapters will be tabulated in this appendix.

1. Uncoupled Motion (equation (81))

$$X = \lambda_1 \tanh \lambda_1 H$$

$$Q_1 = -\frac{2(C_1 a^2)}{(aX)}$$

$$Q_2 = \left(\frac{C_1 a^2}{aX}\right)^2 - \frac{2(C_4 a^3)}{(aX)}$$

$$Q_3 = \frac{(E_2 a^2)}{(aX)} + \frac{(C_1 a^2)}{(aX)}$$

$$Q_4 = (E_1 a) - \frac{(C_1 a^2)}{(aX)}$$

$$Q_5 = \frac{(C_1 a^2)(E_1 a) + (E_7 a^3) + 2(C_4 a^3)}{(aX)}$$

$$Q_6 = (E_6 a^2) - \frac{(C_4 a^3) + (C_1 a^2)(E_1 a)}{(aX)}$$

$$Q_7 = (C_1 a^2) - (E_1 a)(aX)$$

$$Q_8 = -(C_1 a^2) - 2(E_2 a^2)$$

$$\delta_1(t) \approx \frac{a^5}{D(\sinh\lambda_1 H)} \{-(0.001906)(\rho_0 H+\bar{\rho}\bar{h})\dddot{Z}(t)\}$$

where

$$C_1 = -(0.1761747)\lambda_1^2$$

$$C_4 = -(0.0689751)\lambda_1^2 X$$

$$E_1 = (0.3522803)X$$

$$E_2 = (0.1761402)X^2 + 0.0880528\lambda_1^2$$

$$E_6 = 0.2069499\lambda_1^2$$

$$E_7 = 0.5518746\lambda_1^2 X$$

2. Coupled Motion (equations (112) and (113))

$$X = \lambda_1 \tanh\lambda_1 H$$

$$Y = \lambda_2 \tanh\lambda_2 H$$

$$G_1 = -\frac{2(C_1 a^2)}{(aX)} - \frac{2(D_2 a^2)}{(aY)}$$

$$G_2 = -\frac{2(C_2 a^2)}{(aX)}$$

$$G_3 = \left(\frac{C_1 a^2}{aX}\right)^2 + \left(\frac{D_2 a^2}{aY}\right)^2 + \frac{4(C_1 a^2)(D_2 a^2) - 2(C_2 a^2)(D_1 a^2)}{(aX)(aY)} - \frac{2(C_4 a^3)}{(aX)}$$

$$G_4 = \frac{(E_2 a^2)}{(aX)} + \frac{(C_1 a^2)}{(aX)}$$

$$G_5 = (E_1 a) - \frac{(C_1 a^2)}{(aX)} - \frac{2(D_3 a^2)}{(aY)}$$

$$G_6 = (E_3 a) - \frac{(C_2 a^2)}{(aX)}$$

$$G_7 = (E_6 a^2) - \frac{(C_4 a^3) + (C_1 a^2)(E_1 a)}{(aX)} + \frac{(D_1 a^2)(E_4 a) - 2(D_3 a^2)(E_1 a)}{(aY)}$$

$$+ \frac{2(C_1 a^2)(D_3 a^2) - (C_3 a^2)(D_1 a^2)}{(aX)(aY)} + \frac{(D_3 a^2)^2}{(aY)^2}$$

$$G_8 = \frac{(C_3 a^2)}{(aY)} + (E_4 a)\,\frac{(aX)}{(aY)}$$

$$G_9 = \frac{(C_2 a^2)}{(aX)} + \frac{(C_3 a^2) + (E_5 a^2)}{(aY)}$$

$$G_{10} = \frac{(C_1 a^2)(E_1 a) + (E_7 a^3) + 2(C_4 a^3)}{(aX)} + \frac{(D_1 a^2)(E_4 a)}{(aY)}$$

$$+ \frac{2(C_3 a^2)(D_1 a^2) - 2(C_1 a^2)(D_3 a^2) - 2(D_3 a^2)(E_2 a^2) + (D_1 a^2)(E_5 a^2)}{(aX)(aY)}$$

$$G_{11} = - \left[(C_1 a^2) - (E_1 a)(aX) + \frac{2(D_3 a^2)(aX)}{(aY)} \right]$$

$$G_{12} = (C_1 a^2) + 2(E_2 a^2)$$

$$H_1 = - \frac{2(C_1 a^2)}{(aX)} - \frac{2(D_3 a^2)}{(aY)}$$

$$H_2 = \frac{(D_1 a^2)}{(aX)} + \frac{(F_2 a^2)(aY)}{(aX)^2}$$

$$H_3 = \frac{(D_1 a^2)}{(aX)} + \frac{(F_1 a)(aY)}{(aX)}$$

$$H_4 = \frac{(D_2 a^2)}{(aX)} + \frac{(F_3 a)(aY)}{(aX)}$$

$$H_5 = \frac{(D_3 a^3) + (F_2 a^2)(aY) + (D_1 a^2)(F_4 a) - 2(D_3 a^2)(F_1 a)}{(aX)}$$

$$- \frac{(C_1 a^2)(D_1 a^2) + (C_1 a^2)(F_1 a)(aY)}{(aX)^2} - \frac{(D_1 a^2)(D_3 a^2)}{(aX)(aY)}$$

$$H_6 = (F_4 a) - \frac{2(C_1 a^2)}{(aX)} - \frac{(D_3 a^2)}{(aY)}$$

$$H_7 = \frac{(F_5 a^2)}{(aX)} + \frac{(D_2 a^2)}{(aX)} + \frac{(D_3 a^2)}{(aY)}$$

$$H_8 = \frac{(D_1 a^2)(F_4 a) + 2(D_4 a^3)}{(aX)}$$

$$+ \frac{(C_1 a^2)(F_1 a)(aY) + (F_7 a^3)(aY) + (D_1 a^2)(F_5 a^2) - 2(D_3 a^2)(F_2 a^2)}{(aX)^2}$$

$$H_9 = (D_1 a^2) + (F_1 a)(aY)$$

$$H_{10} = \frac{2(F_2 a^3)(aY)}{(aX)} + (D_1 a^2)$$

$$\delta_1(t) \approx - \frac{(0.001906) a^5 (\rho_0 H + \bar{\rho} \bar{h})}{D(\sinh \lambda_1 H)} \dddot{Z}(t)$$

$$\delta_2(t) \approx - \frac{(0.000195) a^5 (\rho_0 H + \bar{\rho} \bar{h})}{D(\sinh \lambda_2 H)} \dddot{Z}(t)$$

where

$$C_1 = - 0.1761747\lambda_1^2$$

$$C_2 = - (0.2661404)\lambda_1^2 + (0.2436875)\lambda_1\lambda_2$$

$$C_3 = - (0.2661404)\lambda_2^2 + (0.2436875)\lambda_1\lambda_2$$

$$C_4 = - (0.0689751)\lambda_1^2 X$$

$$D_1 = - (0.8034958)\lambda_1^2$$

$$D_2 = - (0.3019649)\lambda_1^2 + (0.0824586)\lambda_1\lambda_2$$

$$D_3 = - (0.3019649)\lambda_2^2 + (0.0824586)\lambda_1\lambda_2$$

$$D_4 = - (0.1578072)\lambda_1^2 X$$

$$E_1 = (0.3522803)X$$

$$E_2 = (0.1761402)X^2 + (0.0880528)\lambda_1^2$$

$$E_3 = (0.2661404)X$$

$$E_4 = (0.2661404)Y$$

$$E_5 = (0.2661404)XY + (0.2436875)\lambda_1\lambda_2$$

$$E_6 = (0.2069499)\lambda_1^2$$

$$E_7 = (0.5518746)\lambda_1^2 X$$

$$F_1 = (0.4793192)X$$

$$F_2 = (0.2396596)X^2 - (0.1620883)\lambda_1^2$$

$$F_3 = (0.4793192)X$$

$$F_4 = (0.3019649)Y$$

$$F_5 = (0.3019649)XY + (0.0824586)\lambda_1\lambda_2$$

$$F_6 = 0.1578072)\lambda_1^2$$

$$F_7 = (0.2970843)\lambda_1^2X$$

APPENDIX C

LONGITUDINALLY EXCITED PINNED-END BEAM WITH ITS INITIAL CURVATURE

Let a beam of length ℓ having an initial curvature $w_o(x)$ be subjected to a parametric excitation of the form $P_o + P\cos\Omega t$ (Figure 25, page 97). We assume that the beam is restricted to oscillate in the first mode and that the dynamic deflection and initial curvature are given by

$$w(x,t) = f(t)\ \sin\frac{\pi x}{\ell} \qquad \text{(C-1)}$$

and

$$w_o(x) = f_o\ \sin\frac{\pi x}{\ell} \qquad \text{(C-2)}$$

where $w(x,t)$ is the dynamic deflection measured from the curved axis of the beam. The time function $f(t)$ is found from

$$\ddot{f} + \omega^2(1 - 2\mu\cos\Omega t)f + \alpha f^3 + 2Kf(f\ddot{f} + \dot{f}^2) = \frac{\overline{\omega}^2 f_o}{P_E}(P_o + P\cos\Omega t) \qquad \text{(C-3)}$$

P_E and $\overline{\omega}^2$ denote the Euler buckling load and natural frequency of the first mode, respectively, i.e.,

$$P_E = \frac{\pi^2 EI}{\ell^2} \qquad \text{(C-4)}$$

$$\overline{\omega}^2 = \frac{\pi^4 EI}{m\ell^4} \qquad \text{(C-5)}$$

where m is the mass per unit length. The parameters in the differential equation are

$$\omega^2 = \bar{\omega}^2\left(1 - \frac{P_0}{P_E}\right) \tag{C-6}$$

and

$$\mu = \frac{1}{2}\,\frac{P}{(P_E - P_0)} \tag{C-7}$$

αf^3 is the nonlinear elasticity term and $2Kf(f\ddot{f} + \dot{f}^2)$ is the nonlinear inertia term. The procedures for determining the constants α and K are discussed by Bolotin.[25]

The differential equation (C-3) with nonhomogeneous term is very similar to the nonlinear equation of motion for the uncoupled liquid oscillations in a container with an elastic bottom. This similarity was of considerable value to the problem which we have investigated, since the problem of a parametrically excited beam with initial curvature has been investigated extensively.

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